On the Integrable Structure of Super Yang-Mills Scattering Amplitudes

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Zusammenfassung

Die maximal supersymmetrische Yang-Mills-Theorie im vierdimensionalen Minkowski-Raum, $\mathcal{N} = 4$ SYM, ist ein außergewöhnliches Modell der mathematischen Physik. Dies gilt vor allem im planaren Limes, in dem die Theorie integrabel zu sein scheint. Diese Integrabilität wurde zunächst für das Spektrum der anomalen Dimensionen entdeckt. Inzwischen ist sie auch bei anderen Observablen zu Tage getreten. Insbesondere sind Streuamplituden auf Baumgraphenniveau Invarianten einer Yangschen Algebra, die die superkonforme Algebra $\mathfrak{psu}(2,2|4)$ beinhaltet. Diese unendlichdimensionale Symmetrie ist ein Kennzeichen für Integrabilität. In dieser Dissertation untersuchen wir Verbindungen zwischen solchen Amplituden und integrablen Modellen. Wir verfolgen zweierlei Ziele. Zum einen wollen wir Grundlagen für eine effiziente, auf der Integrabilität basierende Berechnung von Amplituden legen. Zum anderen sind wir bestrebt einen Zugang zu schaffen der neue Ideen zu Amplituden auch für integrable Systeme im Allgemeinen anwendbar macht. Dazu charakterisieren wir Yangsche Invarianten innerhalb der Quanten-Inverse-Streumethode (QISM), die Werkzeuge zur Behandlung integrabler Spinketten bereitstellt. Wir arbeiten mit einer Klasse von Oszillatordarstellungen der Lie-Algebra $\mathfrak{u}(p,q|m)$, die etwa vom Spektralproblem der $\mathcal{N} = 4$ SYM bekannt ist. In diesem Rahmen entwickeln wir Methoden zur Konstruktion von Yangschen Invarianten. Wir zeigen, dass der algebraische Bethe-Ansatz, ein Bestandteil der QISM, für die Erzeugung von Yangschen Invarianten für $\mathfrak{u}(2)$ und im Prinzip auch für $\mathfrak{u}(n)$ verwendet werden kann. Diese Invarianten sind spezielle Zustände inhomogener Spinketten. Die zugehörigen Bethe-Gleichungen lassen sich leicht lösen. Unser Zugang ermöglicht es zudem diese Invarianten als Zustandssummen von Vertexmodellen zu interpretieren. Außerdem führen wir ein unitäres Graßmannsches Matrixmodell zur Konstruktion Yangscher Invarianten mit Oszillatordarstellungen von $\mathfrak{u}(p,q|m)$ ein. Es ist angeregt durch eine Formulierung von Amplituden als mehrdimensionale Konturintegrale auf Graßmannschen Mannigfaltigkeiten. Für einen Spezialfall reduziert sich unsere Formel zu dem Brezin-Gross-Witten-Integral über unitäre Matrizen. Ferner führt es zu einem $U(2)$-Integralausdruck für eine Invariante die einer R-Matrix entspricht. Solche R-Matrizen bilden die Basis für integrable Spinketten. Wir wenden eine auf Bargmann zurückgehende Integraltransformation auf unser unitäres Graßmannsches Matrixmodell an, welche die Oszillatoren in Spinor-Helizitäts-artige Variablen überführt. Dadurch gelangen wir zu einer Weiterentwicklung des bereits erwähnten Graßmannschen Integrals für bestimmte Amplituden. Die maßgeblichen Unterschiede sind, dass wir in der Minkowski-Signatur arbeiten und die Kontur auf die unitäre Gruppenmannigfaltigkeit festgelegt ist. Wir vergleichen durch unser Integral gegebene Yangsche Invarianten für $\mathfrak{u}(2,2|4)$ mit bekannten Ausdrücken für Amplituden und kürzlich eingeführten Deformationen derselben.

Schlagwörter: Super-Yang-Mills-Theorie, Streuamplituden, Graßmannsches Integral, Yangsche Invarianz, Oszillatordarstellungen, integrable Spinketten, Bethe-Ansatz, Vertexmodelle, unitäre Matrixmodelle, Bargmann-Transformation

Abstract

The maximally supersymmetric Yang-Mills theory in four-dimensional Minkowski space, $\mathcal{N} = 4$ SYM, is an exceptional model of mathematical physics. Even more so in the planar limit, where the theory is believed to be integrable. This integrable structure was first revealed for the spectrum of anomalous dimensions. By now it has begun to surface also for further observables. In particular, the tree-level scattering amplitudes were shown to be invariant under the Yangian of the superconformal algebra $\mathfrak{psu}(2,2|4)$. This infinite-dimensional symmetry is a hallmark of integrability. In this dissertation we explore connections between these amplitudes and integrable models. Our aim is twofold. First, we want to lay foundations for an efficient integrability-based computation of amplitudes. Second, we intend to create a formulation that makes new ideas about amplitudes applicable to integrable systems in general. To this end, we characterize Yangian invariants within the quantum inverse scattering method (QISM), which is an extensive toolbox for integrable spin chains. Throughout the thesis we work with a class of oscillator representations of the Lie algebra $\mathfrak{u}(p,q|m)$, that is known e.g. from the $\mathcal{N} = 4$ SYM spectral problem. Making use of this setup, we develop methods for the construction of Yangian invariants. We show that the algebraic Bethe ansatz from the QISM toolbox can be specialized to yield Yangian invariants for $\mathfrak{u}(2)$ and in principle also for $\mathfrak{u}(n)$. These invariants are special states of inhomogeneous spin chains. The associated Bethe equations can be solved easily. Our approach also allows to interpret these Yangian invariants as partition functions of vertex models. What is more, we establish a unitary Graßmannian matrix model for the construction of a subset of $\mathfrak{u}(p,q|m)$ Yangian invariants with oscillator representations. It is inspired by a formulation of amplitudes as multi-dimensional contour integrals on Graßmannian manifolds. In a special case our formula reduces to the Brezin-Gross-Witten integral over unitary matrices. Furthermore, it yields a $U(2)$ integral expression for an invariant corresponding to an R-matrix. Such R-matrices generate integrable spin chain models. We apply an integral transformation due to Bargmann to our unitary Graßmannian matrix model, which turns the oscillators into spinor helicity-like variables. Thereby we are led to a refined version of the aforementioned Graßmannian integral for certain amplitudes. The most decisive differences are that we work in Minkowski signature and that the integration contour is fixed to be a unitary group manifold. We compare $\mathfrak{u}(2,2|4)$ Yangian invariants defined by our integral to known expressions for amplitudes and recently introduced deformations thereof.

Keywords: super Yang-Mills theory, scattering amplitudes, Graßmannian integral, Yangian invariance, oscillator representations, integrable spin chains, Bethe ansatz, vertex models, unitary matrix models, Bargmann transformation

List of Publications

This dissertation is based on the following publications:

[1] R. Frassek, N. Kanning, Y. Ko, and M. Staudacher, "Bethe Ansatz for Yangian Invariants: Towards Super Yang-Mills Scattering Amplitudes," *Nucl. Phys. B* **883** (2014) 373, `arXiv:1312.1693`.

[2] N. Kanning, Y. Ko, and M. Staudacher, "Graßmannian Integrals as Matrix Models for Non-Compact Yangian Invariants," *Nucl. Phys. B* **894** (2015) 407, `arXiv:1412.8476`.

In addition, it contains a considerable amount of unpublished work. The author also contributed to a publication on a closely related subject:

[3] N. Kanning, T. Łukowski, and M. Staudacher, "A Shortcut to General Tree-Level Scattering Amplitudes in N=4 SYM via Integrability," *Fortsch. Phys.* **62** (2014) 556, `arXiv:1403.3382`.

Contents

Chapter 1

Introduction

1.1 Integrable Models

The exact solution of the gravitational two-body problem dates back to Newton's "Principia" published in 1687. It is a landmark in mathematical physics that led to a most profound advance in our physical understanding by establishing the sound theoretical foundation of Kepler's empirical laws of planetary motion. Throughout the centuries this problem has been revisited employing new concepts and machinery developed in the field of classical mechanics. For instance, the relative motion of the two bodies can be derived by exploiting the conserved quantities of the problem, i.e. the energy, the angular momentum vector and the Runge-Lenz vector. In the Hamiltonian formulation this may be achieved by a specific canonical transformation, as reviewed e.g. in [4, 5]. This transformation in particular replaces the three momenta by three of the conserved quantities, which are independent and Poisson-commuting, just as the original momenta. One can choose the energy and the third components of both conserved vectors as these new momenta. The construction of this transformation reduces to integrals and inversions of algebraic equations. In the new coordinates the Hamiltonian equations of motion are solved trivially because the momenta are conserved. The procedure outlined here is a special case of a theorem by Liouville, cf. [6, 7]: The solution of a Hamiltonian system in a $2N$-dimensional phase space with N conserved quantities, which are independent and Poisson-commuting, reduces to a number of integrals and inversions. Therefore such a system is called *integrable* or sometimes synonymously *exactly solvable*.

The gravitational two-body problem is a shining example for the importance of integrable systems. They are mathematically much more accessible than generic models. Their exact solutions can exhibit features that are absent or easily overlooked in approximate methods. Consequently, these solutions can even be critical to settle conceptual questions arising in a new theory. Integrable systems can be found in various branches of physics and there is an opulence of mathematical approaches to investigate them. Naturally, our aim in this introduction can in no way be to provide a comprehensive overview of this fascinating field at the border between physics and mathematics. Thus we concentrate on a selection of examples, which illustrate important themes and are headed towards the subject of this thesis. To compensate for this focus, most of the works we refer to are books or review articles that embed the examples into a broader context. In this section special *emphasis* is put on topics that reappear in later chapters.

The concept of integrability is not limited to the ordinary differential equations of classical mechanics. Let us continue with an example of an integrable partial differential equation appearing in fluid dynamics. Water waves can be modeled by highly non-linear

Euler equations, whose general solution is not known. However, the situation changes radically in a certain limiting case. Restricting to surface waves in shallow water that propagate only in one dimension, one arrives at the Korteweg-de Vries equation. This equation is still non-linear, yet it possesses remarkable hidden structures, as explained e.g. in [8, 9]. For instance, there are solitary wave solutions that can be superposed despite the non-linearity. What is more, the Korteweg-de Vries equation can be formulated as a Hamiltonian system with infinitely many Poisson-commuting conserved quantities. Even the initial value problem for this equation has been solved by means of the so-called inverse scattering method. This method can be understood as a generalization of the Liouville theorem to infinitely many degrees of freedom because in essence it is a canonical transformation to coordinates in which the dynamics becomes linear and in this sense trivial. Thus the Korteweg-de Vries equation and many further partial differential equations solvable by this method are said to be integrable. Let us remark that the Korteweg-de Vries equation also appears in fields that seem to be far removed from fluid dynamics. In particular, classes of solutions can be formulated as *matrix models*, i.e. as integrals over certain matrices, see e.g. [10, 11].

The examples discussed up to this point are models of classical physics. The exact solution of a prototypical quantum integrable model was obtained by Bethe already shortly after the advent of quantum mechanics in 1931. He studied a one-dimensional chain of electron spins described by Pauli matrices, which Heisenberg had proposed earlier as a model of a ferromagnet. Bethe made an ansatz for the wave functions. He showed that for this ansatz to yield eigenfunctions of the Hamilton operator, its parameters have to obey a set of algebraic equations. This resulted in an efficient method for the diagonalization of the Hamiltonian. Detailed accounts on this *Bethe ansatz* are provided e.g. in [12, 13].

Bethe's solution of the Heisenberg spin chain contains no direct reference to the notion of integrability we encountered in the previous paragraphs. This link was established only several decades later by Faddeev and his coworkers in the context of what they termed *quantum inverse scattering method* (QISM) [14, 15]. At the core of this approach lies the cubic *Yang-Baxter equation*, see [16] for a compilation of pioneering publications. The group around Faddeev managed to express the Hamiltonian of the Heisenberg spin chain and its eigenfunctions in terms of a particular solution of the Yang-Baxter equation and thereby incorporated the Bethe ansatz into their framework. This reformulation immediately led to a family of commuting operators that contains the Hamiltonian. Therefore these operators are analogous to the Poisson-commuting conserved quantities in classically integrable models. In addition, the reparameterization of the model from Pauli matrices to a solution of the Yang-Baxter equation allows for an interpretation as the counterpart of the canonical transformation that is at the heart of the Liouville theorem and the inverse scattering method, cf. [17]. To this effect, the QISM is a quantization of these classical results.

A significant feature of the QISM is that it can be extended to a large class of *integrable spin chains* by choosing different solutions of the Yang-Baxter equation. Many of these solutions have a representation theoretic origin. From this perspective an integrable spin chain model is specified by the choice of a symmetry algebra and a representation thereof. The Heisenberg model arises from the Lie algebra $\mathfrak{su}(2)$ with the spin $\frac{1}{2}$ representation. In fact, a closer look reveals that this Lie algebra is extended to an infinite-dimensional symmetry algebra referred to as *Yangian*, see e.g. [18, 19, 20]. Such hidden extended symmetries are characteristic of integrable models. The QISM has found applications beyond spin chain models. It is of utility for two-dimensional *vertex models* in statistical physics, where solutions of the Yang-Baxter equation provide the Boltzmann weights at the vertices, see also [21]. Furthermore, it applies to certain $1+1$-dimensional integrable quantum field theories, in which the Yang-Baxter equation characterizes a factorization of the scattering matrix into a succession of two-particle scattering events. However, most applications of the QISM are restricted to such low-dimensional theories.

The world we experience is $3 + 1$-dimensional. The standard model of particle physics forms the foundation of our current physical understanding of this world at a subatomic level. It incorporates the electromagnetic interactions along with the weak and the strong nuclear force. Its theoretical predictions are in impressive agreement with experimental results. Nevertheless, most of our knowledge about this model is based on perturbative calculations rather than exact results. Especially in quantum chromodynamics (QCD), the theory of the strong nuclear force, there is a strongly coupled regime that is hardly accessible by such methods.

Can one use ideas from integrable models to gain deeper insights into QCD? The situation is in some sense similar to that of water waves discussed above. There are certain limiting cases of QCD that can be mapped to integrable models. An example is provided by the scattering of two hadrons in the multicolor limit, which is also called planar limit for reasons that will be explained in the following section, and Regge kinematics, i.e. at high energies and fixed momentum transfer. This system can be described in terms of a spin chain that is exactly solvable by a variant of the QISM [22, 23]. It is closely related to the Heisenberg model, the essential difference being the replacement of the spin $\frac{1}{2}$ representation of $\mathfrak{su}(2)$ by an infinite-dimensional representation of $\mathfrak{sl}(\mathbb{C}^2)$. A readable discussion of this example of integrability in QCD may be found in [24], see also the extensive review [25].

Despite these achievements, an exact solution of the complete Yang-Mills dynamics of QCD, even in the planar case, seems to be out of reach, if it exists at all. In fact, to this day no $3 + 1$-dimensional interacting quantum field theory has been solved exactly. However, this unsatisfactory situation may change in the foreseeable future as an integrable structure has began to surface in a supersymmetric relative of QCD during the past 15 years. By now there is overwhelming evidence that *planar maximally supersymmetric Yang-Mills theory*, for short planar $\mathcal{N} = 4$ SYM, is integrable [26]. Like Newton's solution of the two-body problem centuries ago, the integrability of this model could prove to be a blueprint for aspects of current fundamental physics. It might contribute to the mathematical underpinnings of quantum field theories in general and advance our understanding of QCD in particular. These prospects constitute our motivation for the investigation of the quantum integrable structure of this planar $\mathcal{N} = 4$ model in the thesis at hand.

1.2 Planar $\mathcal{N} = 4$ Super Yang-Mills Theory

The Lagrangian of pure Yang-Mills theory in four-dimensional Minkowski space solely comprises bosonic gauge fields. To obtain a supersymmetric theory, these have to be supplemented by further fields, in particular fermionic ones. The number of supersymmetry transformations is characterized by a positive integer $\mathcal{N}$. It cannot be larger than four in order to be able to avoid gravitational degrees of freedom, which makes $\mathcal{N} = 4$ SYM the *maximally supersymmetric Yang-Mills theory* in four dimensions. This theory was introduced in [27, 28], see also the reviews [29, 30] and some historical recollections in [31]. It is essentially specified by only three parameters: the coupling constant g_{YM}, the number of colors N_{C} of the gauge group $SU(N_{\mathrm{C}})$ and the instanton angle θ_{I}. In this model the gauge bosons in the Lagrangian are accompanied by scalar fields and fermions, which are all $N_{\mathrm{C}} \times N_{\mathrm{C}}$ matrices in color space, i.e. they transform in the adjoint representation of the gauge group. The details of this field content are dictated by the representation theory of the supersymmetry algebra. In fact, beyond that the fields transform in a representation of the *superconformal algebra* $\mathfrak{psu}(2,2|\mathcal{N}=4)$. This Lie superalgebra contains the conformal algebra $\mathfrak{su}(2,2) \simeq \mathfrak{so}(4,2)$ as well as the internal R-symmetry algebra $\mathfrak{su}(\mathcal{N}=4)$. Let us interject here that bosonic Yang-Mills theory is conformally invariant as a classical field theory. However, this symmetry is broken at the quantum level. Remarkably, this changes in the supersymmetric model. $\mathcal{N}=4$ SYM is a superconformal *quantum* field theory. This suggests that, in spite of the larger field content, the supersymmetric theory is in effect simpler than its bosonic counterpart.

We already mentioned one motivation for the investigation of $\mathcal{N}=4$ SYM in the previous section. Although it is not realized in nature, it may serve as a mathematical toy model for more realistic theories like QCD. Further impetus comes from the *AdS/CFT correspondence* proposed in 1997 [32, 33, 34], see also the rather recent review [35] and the references to more comprehensive treatises therein. It conjectures the equivalence of certain string theories on backgrounds containing an anti-de Sitter (AdS) space and conventional quantum field theories with conformal symmetry (CFT) that are basically defined on the boundary of that space. The most thoroughly studied example of this correspondence relates type IIB superstrings on the product of five-dimensional AdS space and a five-sphere to $\mathcal{N}=4$ SYM on four-dimensional Minkowski space. As a plausibility check, let us mention that the isometries of the string background match the superconformal symmetry of the quantum field theory as they are both described by the algebra $\mathfrak{psu}(2,2|4)$. A feature which makes the correspondence between the two different types of models particularly interesting, and at the same time hard to prove, is its strong/weak type. Strongly coupled regimes in $\mathcal{N}=4$ SYM are related to weakly coupled string theory and therefore accessible by perturbative string calculations, and vice versa. Initial evidence for the AdS/CFT correspondence was found in the 't Hooft limit of $\mathcal{N}=4$ SYM, where $N_\mathrm{C} \to \infty$, the coupling $g_\mathrm{YM}^2 N_\mathrm{C}$ is kept at a fixed value and θ_I is believed to be irrelevant. This is also referred to as *planar limit* because in this case only Feynman graphs without intersections contribute in the perturbative expansion of the SYM theory. On the string side of the correspondence this translates into the limit of free strings. As both theories simplify considerably in this limit, it is where most precision tests of the AdS/CFT correspondence have been performed. In the past almost two decades the correspondence has received the attention of many scientists, which resulted in considerable progress. Despite these efforts, a proof is still missing.

The simplicity of $\mathcal{N}=4$ SYM in the planar limit is closely connected to an underlying *integrable structure*, whose emergence is detailed in the comprehensive reviews series [26], see e.g. [36] for a more concise presentation. These developments enabled crucial tests of the AdS/CFT correspondence for this theory. One might even argue that a complete understanding of the integrable structure is a prerequisite for its proof in the planar limit. The evidence of integrability in planar $\mathcal{N}=4$ SYM is rooted in the concepts discussed in the previous section 1.1. A fertile perspective is to view integrability as an infinite-dimensional extension of the superconformal symmetry algebra $\mathfrak{psu}(2,2|4)$ that should determine all observables of the quantum field theory. For important classes of observables, and often to a certain order in perturbation theory, this point of view was worked out in detail and the integrability is proven. In countless further cases the predictions of assuming integrability were verified by laborious quantum field theory calculations. These accumulated results point towards the integrability of planar $\mathcal{N}=4$ SYM. Yet at present, the origin of this extraordinary structure remains mostly a mystery. Unraveling it will not only lead to the exact solution of this one $3+1$-dimensional quantum field theory but likely go along with new insights into integrable models in general.

The first class of observables that was related to an integrable system are the anomalous dimensions of local gauge invariant operators, which determine the two-point functions of these operators. At one loop in the perturbative expansion it was proven that they can be described by an integrable spin chain [37, 38, 39, 40], see also [41] of the review series mentioned above. The aforesaid operators make up the states of this spin chain. They consist of a color trace over a number of fields and derivatives thereof at a single spacetime point. The fields in the operator correspond to the sites of the spin chain. These sites transform in an infinite-dimensional representation of the superconformal algebra $\mathfrak{psu}(2,2|4)$, which replaces the spin $\frac{1}{2}$ representation of $\mathfrak{su}(2)$ from the Heisenberg spin chain of section 1.1. Scale transformations of planar $\mathcal{N} = 4$ SYM are generated by a dilatation operator that acts on the spin chain states. This dilatation operator receives quantum corrections. At one loop these can be identified with the Hamiltonian of an integrable $\mathfrak{psu}(2,2|4)$ spin chain, which is a straightforward generalization of the Heisenberg Hamiltonian and contains only interactions of neighboring sites. Therefore the spectrum of these quantum corrections, the so-called set of anomalous dimensions, agrees with the energy spectrum of the integrable spin chain. Consequently, a Bethe ansatz can be applied to compute the anomalous dimensions. This method circumvents standard Feynman graph calculations and thereby significantly simplifies the computation of the anomalous dimensions, i.e. the solution of the *spectral problem*. Lastly, let us note how this solution relates to the perspective put forward in the previous paragraph. The spin chain of the one-loop spectral problem is based on an infinite-dimensional algebra, the Yangian of $\mathfrak{psu}(2,2|4)$. The role of this Yangian was also emphasized in [42, 43].

Strong evidence for the integrability of the spectral problem persists beyond one loop, where the structure becomes much more intricate. The range of the interactions in the dilatation operator increases with the loop order. The explicit form of this operator beyond one loop is only known for some subsectors of the full $\mathfrak{psu}(2,2|4)$ symmetric field content. In such subsectors it can be mapped to long-range integrable spin chains. However, this description neglects wrapping effects, which occur if the range of interaction equals or exceeds to the length of the spin chain. What is more, neglecting these effects it is even possible to formulate asymptotic all-loop Bethe equations. They are based on a choice of a vacuum state which reduces the $\mathfrak{psu}(2,2|4)$ symmetry to a residual $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra at one-loop level. The all-loop result can then be obtained by encoding the coupling constant into a central extension of this residual algebra. These developments are summarized in [44, 45, 46] of the review series.[1] There are even proposals for the complete all-loop spectrum of anomalous dimensions including wrapping. The most elaborate one is a system of equations called "quantum spectral curve" [47, 48]. Its predictions have passed important tests. Notwithstanding these impressive achievements, the all-loop dilatation operator itself, whose spectrum those equations are believed to describe, remains unknown. Consequently, also the infinite-dimensional symmetry algebra generalizing the Yangian of $\mathfrak{psu}(2,2|4)$ at one loop to finite values of the coupling has not yet been revealed.[2]

If planar $\mathcal{N} = 4$ SYM is integrable, this should manifest itself not only in the spectrum of anomalous dimensions, which is closely related to two-point functions, but also for further observables. A natural step is to investigate the integrability of three-point functions, see e.g. the recent approach [51] and the list of references therein. Another class of observables are *scattering amplitudes*, where in particular at tree-level a Yangian symmetry has been

[1] Here we merely sketched the developments on the $\mathcal{N} = 4$ SYM side of the AdS/CFT correspondence. These were paralleled by the discovery of an integrable structure on the string side, namely at the classical level the string theory is described by an integrable sigma model, see once again the review series [26].

[2] At this point it is worth noting some recent work on a Yangian structure, or rather more generally a quantum group structure, at the level of the centrally extended residual symmetry algebra [49, 50].

exposed [52]. It is the very same Yangian of $\mathfrak{psu}(2,2|4)$ that also features in the one-loop spectral problem. This connection might shed light on a uniform integrable structure underlying the whole of planar $\mathcal{N}=4$ SYM. For this reason we focus on the integrability of scattering amplitudes in the present thesis.

In the subsequent section 1.3 we provide an introduction to tree-level amplitudes of $\mathcal{N}=4$ SYM emphasizing their integrable structure. The technical level is such that it covers all the concepts and formulas needed later on. It also enables us to formulate in detail the objectives and the outline of this thesis in section 1.4.

1.3 Super Yang-Mills Scattering Amplitudes

In this section, we present a brief review of planar $\mathcal{N}=4$ SYM scattering amplitudes. After introducing significant fundamentals, we focus on those results and techniques that form the basis for the original research presented later in this thesis. In particular, we discuss the integrable structure of these amplitudes and recently proposed deformations thereof, which preserve integrability. These deformations are a step towards using powerful integrability techniques for the efficient computation of scattering amplitudes. Broader discussions of scattering amplitudes in gauge theories can be found in the recent books [53, 54] and e.g. in the concise review article [55]. The intriguing features of amplitudes in the planar $\mathcal{N}=4$ model are highlighted in the reviews [56, 57]. Except for the recently introduced deformations, the topics covered in this section are discussed in these texts. In addition, we provide references to the original literature for key results and in case we find the exposition particularly instructive.

1.3.1 Color-Decomposition and Spinor Helicity Variables

Before we can present formulas for $\mathcal{N}=4$ SYM scattering amplitudes and discuss their properties, such as symmetries, we have to organize the amplitudes in a way that makes these features most accessible. This is achieved by stripping off the color structure from the gauge theory amplitudes by means of a color decomposition. The resulting partial amplitudes depend on the kinematics, i.e. the particle momenta. They allow for a perturbative expansion and we restrict our discussion for the most part to the tree-level contribution. Furthermore, to obtain manageable formulas for these tree-level partial amplitudes a clever parameterization of the momenta is of importance. We choose to express the momenta using spinor helicity variables. In what follows, we explain these techniques in more detail.

Let us first discuss the *color decomposition* of perturbative gauge theory scattering amplitudes, see [58] and references therein. The model of interest for us is four-dimensional $\mathcal{N}=4$ SYM with gauge group $SU(N_\text{C})$. In a scattering process in this theory each of the participating massless particles $i=1,\ldots,N$ is associated with a momentum null vector $p^i\in\mathbb{R}^{1,3}$, a helicity $h^i=-1,-\frac{1}{2},0,+\frac{1}{2},+1$, and a color index $a^i=1,\ldots,{N_\text{C}}^2-1$. Additional internal R-symmetry quantum numbers of the particles are not important for the color-decomposition and therefore suppressed here. The full gauge theory scattering amplitude $\mathscr{A}_N(p^1,h^1,a^1;\ldots;p^N,h^N,a^N)\equiv\mathscr{A}_N(\{p^i,h^i,a^i\})$ can be expressed in terms of *partial amplitudes* $A_N(\{p^i,h^i\})$, also called *color-stripped amplitudes*, that are independent of the color indices,

$$\mathscr{A}_N(\{p^i,h^i,a^i\})=g_\text{YM}^{N-2}\sum_{\sigma\in S_N/\mathbb{Z}_N}\operatorname{tr}(T_{a^{\sigma(1)}}\cdots T_{a^{\sigma(N)}})A_N(\{p^{\sigma(i)},h^{\sigma(i)}\})+\begin{matrix}\text{multi-trace}\\\text{terms}\end{matrix}. \quad (1.1)$$

Hence this procedure allows to separate the color structure of the amplitude from the kinematics. Here g_{YM} denotes the Yang-Mills coupling constant. Moreover, T_a are Hermitian traceless $N_{\mathrm{C}} \times N_{\mathrm{C}}$ matrices, which are normalized such that $\mathrm{tr}(T_a T_b) = \delta_{ab}$. The summation in this formula runs over all $(N-1)!$ non-cyclic permutations σ of N elements. We suppressed the explicit form of terms involving products of multiple color traces. These terms do not contribute in the planar limit of the theory, where $N_{\mathrm{C}} \to \infty$ while the 't Hooft coupling $g^2_{\mathrm{YM}} N_{\mathrm{C}}$ is kept constant.

The partial amplitudes have a perturbative expansion in the 't Hooft coupling, see e.g. the exposition in [59]. Schematically, we may write

$$A_N = A_N^{(\mathrm{tree})} + \sum_{L=1}^{\infty} \left(\frac{g^2_{\mathrm{YM}} N_{\mathrm{C}}}{8\pi^2} \right)^L A_N^{(L)} \,. \tag{1.2}$$

The L-loop partial amplitude $A_N^{(L)}$ suffers from infrared singularities that are commonly dealt with using dimensional regularization. From now on we focus on the *tree-level partial amplitude* $A_N^{(\mathrm{tree})}$. The aforementioned review articles [56, 57] also cover loop amplitudes. Let us remark that at tree-level the multi-trace terms in (1.1) are absent even in the non-planar theory. In slight abuse of terminology, we sometimes refer to $A_N^{(\mathrm{tree})}$ simply as "tree-level amplitude" or even just as "amplitude". As we will review shortly, already this tree-level contribution displays some remarkable properties that are deeply related to the integrable structure of planar $\mathcal{N} = 4$ SYM.

After eliminating the color structure, we can focus on the kinematics encoded in the partial amplitudes. For this purpose the choice of *spinor helicity variables* [60, 61, 62, 63, 64, 65] for the particle momenta is most appropriate because it leads to particularly simple expressions for the amplitudes. To introduce these variables we use a bijection between Minkowski space $\mathbb{R}^{1,3}$ and the space of Hermitian 2×2 matrices. A Minkowski vector $p = (p_\mu)$ is represented by the matrix

$$(p_{\alpha\dot\beta}) = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} , \tag{1.3}$$

where the indices take the values $\alpha, \dot\beta = 1, 2$. Using the Minkowski inner product $p \cdot q = p_0\, q_0 - \vec{p} \cdot \vec{q}$ one verifies that $\det(p_{\alpha\dot\beta}) = p^2$. For the scattering of massless particles we are dealing with null momenta, $p^2 = 0$. Hence the corresponding matrix is at most of rank 1 and can thus be expressed as

$$p_{\alpha\dot\beta} = \lambda_\alpha \tilde\lambda_{\dot\beta} \tag{1.4}$$

with two spinors $\lambda = (\lambda_\alpha), \tilde\lambda = (\tilde\lambda_{\dot\beta}) \in \mathbb{C}^2$. Imposing this matrix to be Hermitian restricts $\tilde\lambda$ to be a real multiple of the complex conjugate spinor $\overline{\lambda}$. Furthermore, (1.4) is invariant under the rescaling $\lambda \mapsto z\lambda$ and $\tilde\lambda \mapsto z^{-1}\tilde\lambda$ with $z \in \mathbb{C}$. This allows us to restrict to spinors satisfying the reality condition

$$\tilde\lambda = \pm\overline{\lambda} \,. \tag{1.5}$$

The sign determines the sign of the energy $\mathrm{sgn}(p_0)$ of the null momentum as $\pm 2p_0 = |\lambda_1|^2 + |\lambda_2|^2$. In the field of scattering amplitudes one often works with complexified momenta, i.e. independent spinors λ and $\tilde\lambda$ that do *not* obey the reality condition (1.5). While we adapt to this habit in some parts of the present introduction, the condition will be of crucial importance later on in this thesis. Let us also mention that the helicity h^i of particle i in a scattering process can be measured by applying a differential operator in the spinor variables,

$$\mathfrak{h}^i A_N^{(\text{tree})}(\{p^i, h^i\}) = h^i A_N^{(\text{tree})}(\{p^i, h^i\}), \tag{1.6}$$

where

$$\mathfrak{h}^i = \frac{1}{2}\bigg(-\sum_{\alpha=1}^{2} \lambda_\alpha^i \partial_{\lambda_\alpha^i} + \sum_{\dot\alpha=1}^{2} \tilde\lambda_{\dot\alpha}^i \partial_{\tilde\lambda_{\dot\alpha}^i} \bigg). \tag{1.7}$$

The partial amplitudes $A_N^{(\text{tree})}$ are conveniently expressed in terms of the *angle* and *square spinor brackets*

$$\langle ij\rangle = \lambda_1^i\lambda_2^j - \lambda_2^i\lambda_1^j, \quad [ij] = -\tilde\lambda_1^i\tilde\lambda_2^j + \tilde\lambda_2^i\tilde\lambda_1^j, \tag{1.8}$$

respectively. Up to a phase, these can be thought of as square roots of the Mandelstam variable

$$s_{ij} = (p^i + p^j)^2 = \langle ij\rangle[ji] \tag{1.9}$$

for null momenta p^i, p^j. It is important to manipulate these brackets efficiently. Therefore we state some of their properties. Making use of (1.5), one obtains

$$[ij] = -\operatorname{sgn}(p_0^i)\operatorname{sgn}(p_0^j)\overline{\langle ij\rangle}. \tag{1.10}$$

From (1.8) the antisymmetry is immediate,

$$\langle ij\rangle = -\langle ji\rangle, \quad [ij] = -[ji]. \tag{1.11}$$

Furthermore, we have the *Schouten identity*

$$\langle ij\rangle\langle kl\rangle - \langle ik\rangle\langle jl\rangle = \langle il\rangle\langle kj\rangle, \tag{1.12}$$

which also holds for square brackets. In the considered scattering processes of N massless particles with momenta p^i the total momentum P is conserved. This condition,

$$P = \sum_{i=1}^{N} p^i = 0, \quad \text{reads} \quad \sum_{i=1}^{N}\langle ki\rangle[il] = 0 \tag{1.13}$$

for all $k, l = 1, \ldots, N$ when expressed in terms of spinor brackets. We stress that for momentum conservation to hold both signs in (1.5) are needed, i.e. there have to be particles with positive and negative energy.

Besides the spinor helicity variables discussed here, also twistors [66] or rather super-twistors are well suited for the study of scattering amplitudes in $\mathcal{N} = 4$ SYM. They have been intensively investigated after featuring prominently in [67]. Let us also mention the momentum twistor variables introduced in [68]. In this work we keep hold of the spinor helicity variables mainly for two reasons. First, it is easy to work with real momenta by imposing (1.5). What is more, while these variables are associated with the conformal algebra $\mathfrak{su}(2,2)$ of four-dimensional Minkowski space, they naturally generalize to certain oscillator representations of superalgebras $\mathfrak{su}(p,q|m)$. These representations will play an important role in this thesis.

1.3.2 Gluon Amplitudes

At this point everything is set up to present expressions for tree-level partial amplitudes $A_N^{(\text{tree})}$. While in $\mathcal{N}=4$ SYM each particle can have a helicity $h^i=-1,-\frac{1}{2},0,+\frac{1}{2},+1$, let us for the moment restrict to amplitudes with $h^i=\pm 1$, i.e. scattering of positive and negative helicity gluons, which we denote by $g^i_\pm$. These tree-level *gluon amplitudes* of the $\mathcal{N}=4$ model are identical to those in QCD. Hence they are even of phenomenological importance and were studied intensively already in the 1980s. However, the textbook approach to the computation of scattering amplitudes via Feynman diagrams quickly reaches its limit. The number of diagrams contributing grows very fast with the number of gluons N, see e.g. the discussion in [69]. Moreover, the computation of individual Feynman diagrams completely obscures a remarkable simplicity of the expressions for the complete amplitudes. Therefore alternative techniques have been developed. Rather than explaining these approaches in detail, in this section we merely state their output: very handy expressions for amplitudes that are useful numerically as well as analytically.

The total momentum in a scattering event is conserved, $P=\sum_i p^i=0$. However, the total helicity $H=\sum_i h^i$ is not. If K is the number of negative helicity gluons, then $H=N-2K$. It turns out to be helpful to classify the gluon amplitudes by the degree of helicity violation, see also figure 1.1. Therefore we denote them by $A^{(\text{tree})}_{N,K}(\{p^i,\pm 1\})\equiv A^{(\text{tree})}_{N,K}(\{g^i_\pm\})$, where we added K as a subscript. One finds that amplitudes with no or one gluon of negative helicity vanish,

$$\begin{aligned} A^{(\text{tree})}_{N,0}(g^1_+,\dots,g^N_+)&=0\,,\\ A^{(\text{tree})}_{N,1}(g^1_+,\dots,g^i_-,\dots,g^N_+)&=0 \quad \text{for} \quad N\geq 3\,. \end{aligned} \tag{1.14}$$

The same holds true for amplitudes with no or one positive helicity gluon. The first non-trivial amplitudes are those with two negative helicity gluons. Hence these are called *maximally helicity violating* (MHV) *amplitudes*. They are given by the exceedingly simple formula[3]

$$A^{(\text{tree})}_{N,2}(g^1_+,\dots,g^i_-,\dots,g^j_-,\dots,g^N_+)=\frac{\delta^4(P)\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle N-1\,N\rangle\langle N1\rangle}\,, \tag{1.15}$$

where $N\geq 4$ and the momentum conservation is implemented by[4]

$$\delta^4(P)=\frac{1}{2}\prod_{\mu=0}^{3}\delta(P_\mu)=\delta(P_{11})\delta(P_{22})\delta(\operatorname{Re}P_{21})\delta(\operatorname{Im}P_{21}) \quad \text{with} \quad P_{\alpha\dot\beta}=\sum_{i=1}^{N}\lambda^i_\alpha\tilde\lambda^i_{\dot\beta}\,. \tag{1.16}$$

Amplitudes with exactly two gluons of positive helicity, so-called $\overline{\text{MHV}}$ amplitudes, are given by the complex conjugate of (1.15). The *Parke-Taylor formula* (1.15) was conjectured in [70] and proven in [71] by recursion and employing gluons that are off the mass shell. It is worth highlighting the simplicity of this formula by bringing the earlier work [72] of Parke and Taylor to our attention. This article was submitted only a few months before [70] and contains the result of a clever Feynman diagrammatic calculation for the MHV amplitude with $N=6$. However, in stark contrast to the single term in (1.15), the result spreads over multiple pages. Nevertheless, this result was considered useful for numerical evaluation

[3] We do not keep track of overall numerical prefactors of amplitudes in this introductory chapter because we are primarily interested in symmetries that are determined by the functional dependence on the momenta, see section 1.3.4.

[4] Here we included the numerical prefactor $\frac{1}{2}$ in order to conform with our conventions in later chapters.

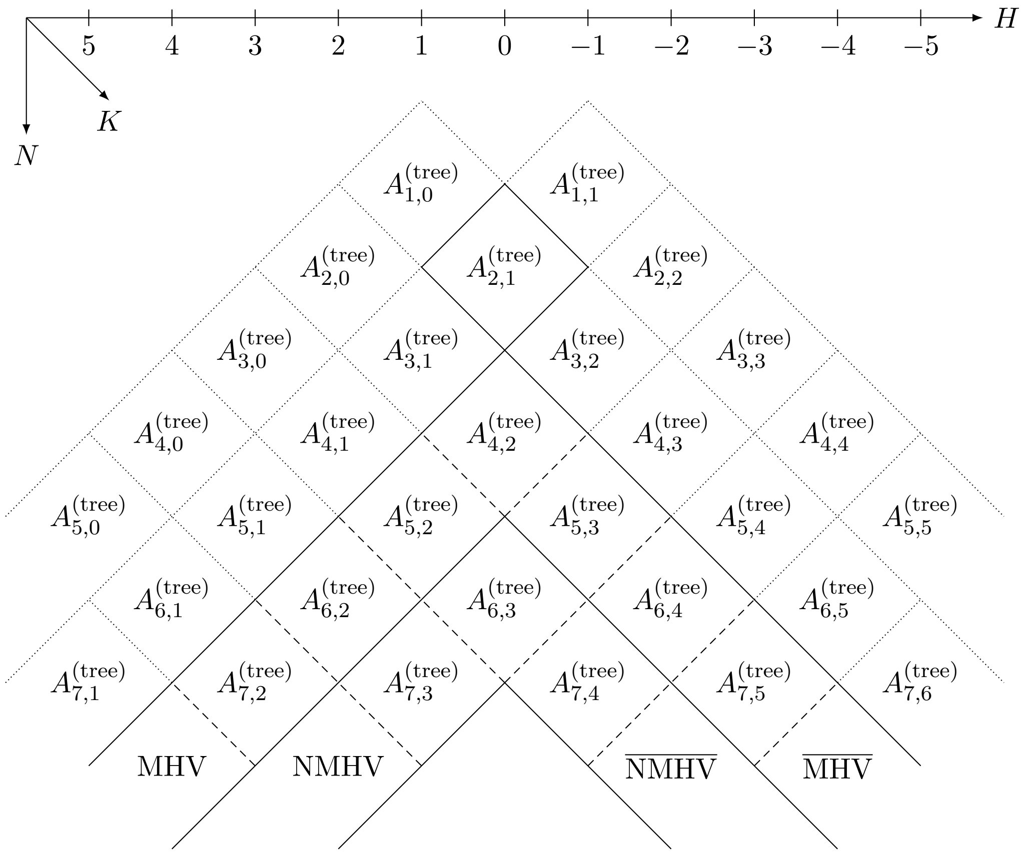


Figure 1.1: Tree-level partial amplitudes $A^{(\text{tree})}_{N,K}$ with N gluons out of which K have negative helicity and with total helicity $H = \sum_i h^i = N - 2K$ are classified by the degree of helicity violation, the so-called MHV degree. This leads to two series of amplitudes: N^{K-2}MHV for $H \geq 0$ and $\overline{\text{N}^{N-K-2}\text{MHV}}$ for $H \leq 0$. Amplitudes enclosed by dotted lines vanish. The amplitude $A^{(\text{tree})}_{2,1}$ can be thought of as free propagation of a gluon. The classification carries over to superamplitudes $\mathcal{A}^{(\text{tree})}_{N,K}$ discussed in section 1.3.3.

and therefore an experimentalist's delight. To quote [72]: "Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight." Indeed, this was achieved with (1.15). By hindsight, one might be tempted to attribute the simplicity of (1.15) to some underlying hidden symmetry like an integrable structure. Historically however, the connection between amplitudes and integrability was established differently, as we will discuss in section 1.3.4.

We continue the classification of amplitudes beyond the MHV case. Amplitudes with $N \geq 6$ gluons out of which $K = 3$ have negative helicity are called *Next-to*-MHV (NMHV). An example with a particular distribution of the negative helicity gluons is

$$\begin{aligned} &A^{(\text{tree})}_{6,3}(g^1_+, g^2_+, g^3_+, g^4_-, g^5_-, g^6_-) \\ &\qquad = \frac{\delta^4(P)}{[5|1+6|2\rangle}\left(\frac{\langle 6|1+2|3]^3}{\langle 61\rangle\langle 12\rangle[34][45]s_{126}} + \frac{\langle 4|5+6|1]^3}{\langle 23\rangle\langle 34\rangle[16][65]s_{156}}\right). \end{aligned} \tag{1.17}$$

Here we used the shorthand notation $[5|1+6|2\rangle = [51]\langle 12\rangle + [56]\langle 62\rangle$ etc. and we defined the variable

$$s_{ijk} = (p^i + p^j + p^k)^2 = \langle ij\rangle[ji] + \langle ik\rangle[ki] + \langle jk\rangle[kj] \tag{1.18}$$

generalizing (1.9). The two-term expression in (1.17) for this amplitude was obtained in [73] as a limit of a seven-gluon amplitude. A slightly more complicated expression containing three terms is known much longer [74].

To complete the classification, one refers to amplitudes with N gluons out of which K have negative helicity and which have total helicity $H = N - 2K \geq 0$ as *Next-to-*$(K-2)$-MHV or, for short N^{K-2}MHV. For $H \leq 0$ these amplitudes belong to the $\overline{\mathrm{N}^{N-K-2}\mathrm{MHV}}$ series. This classification of amplitudes is summarized in figure 1.1. The complexity of the expressions for the amplitudes tends to increase with an decreasing degree of helicity violation. This is illustrated by comparing the single-term formula (1.15) for the MHV amplitudes with the two-term expression for the NMHV amplitude in (1.17). Let us already mention that helicity conserving amplitudes with $H = 0$ will be of special interest later in this thesis.

So far we in essence just *presented* formulas for some sample amplitudes. Let us also briefly mention the *Britto-Cachazo-Feng-Witten* (BCFW) *on-shell recursion relations* [75, 76], that can be used to *construct* tree-level amplitudes. This method makes use of the analytic properties of the amplitudes as functions of the complexified particle momenta. An amplitude is constructed iteratively from multiple amplitudes with less particles. In particular, with the BCFW recursion one recovers the sample amplitudes (1.15) and (1.17) presented in this section.

1.3.3 Superamplitudes

After concentrating on gluon scattering in the previous section, we extend this discussion to the complete particle content of $\mathcal{N} = 4$ SYM. It can be deduced from the representation theory of the superconformal algebra $\mathfrak{psu}(2, 2|4)$ and is summarized by:

particles	h
1 positive helicity gluon g_+	$+1$
4 gluinos $\tilde{g}_{\dot{a}}$	$+\frac{1}{2}$
6 scalars $\varphi_{\dot{a}\dot{b}} = -\varphi_{\dot{b}\dot{a}}$	0
4 antigluinos $\bar{\tilde{g}}_{\dot{a}}$	$-\frac{1}{2}$
1 negative helicity gluon g_-	-1

(1.19)

Here the indices[5] $\dot{a}, \dot{b} = 1, \ldots, 4$ are associated with the internal $\mathfrak{su}(4)$ R-symmetry of the model. Furthermore, we display the helicity h of each particle because as in the previous section it is important to classify the amplitudes. Instead of discussing the scattering amplitudes of the particles listed in (1.19) individually, it is convenient to package them into *superamplitudes*. For this the particles are organized into a superfield [77],

$$\begin{aligned}\Phi = g_+ + \sum_{\dot{a}} \eta_{\dot{a}}\, \tilde{g}_{\dot{a}} + \frac{1}{2!}\sum_{\dot{a},\dot{b}} \eta_{\dot{a}}\eta_{\dot{b}}\, \varphi_{\dot{a}\dot{b}} \\ + \frac{1}{3!}\sum_{\dot{a},\dot{b},\dot{c},\dot{d}} \eta_{\dot{a}}\eta_{\dot{b}}\eta_{\dot{c}}\, \epsilon_{\dot{a}\dot{b}\dot{c}\dot{d}}\, \bar{\tilde{g}}_{\dot{d}} + \frac{1}{4!}\sum_{\dot{a},\dot{b},\dot{c},\dot{d}} \eta_{\dot{a}}\eta_{\dot{b}}\eta_{\dot{c}}\eta_{\dot{d}}\, \epsilon_{\dot{a}\dot{b}\dot{c}\dot{d}}\, g_- \,,\end{aligned} \tag{1.20}$$

[5] A reader familiar with the field of $\mathcal{N} = 4$ SYM scattering amplitudes would probably expect undotted indices a, b here. We use $\dot{a}, \dot{b}$ in order to comply with conventions that are natural in later chapters of this thesis. Throughout the present introductory chapter this is the main principle for selecting our notation. In particular, we also place the indices of $\bar{\tilde{g}}_{\dot{a}}$ and $\eta_{\dot{a}}$, see (1.20) below, downstairs for this reason.

where the completely antisymmetric symbol is fixed by $\epsilon_{1234} = 1$ and we introduced the Graßmann variables $\eta_{\dot{a}}$ obeying $\eta_{\dot{a}}\eta_{\dot{b}} = -\eta_{\dot{b}}\eta_{\dot{a}}$. Extending the definition of the helicity operator in (1.7), we define a “superhelicity” operator that has the eigenvalue 1 when acting on the superfield (1.20),

$$\frac{1}{2}\bigg(- \sum_{\alpha=1}^{2} \lambda_\alpha \partial_{\lambda_\alpha} + \sum_{\dot{\alpha}=1}^{2} \tilde{\lambda}_{\dot{\alpha}} \partial_{\tilde{\lambda}_{\dot{\alpha}}} + \sum_{\dot{a}=1}^{4} \eta_{\dot{a}} \partial_{\eta_{\dot{a}}} \bigg) \Phi = 1\, \Phi \,. \tag{1.21}$$

Superamplitudes $\mathcal{A}_N^{(\text{tree})}$ can be understood as scattering amplitudes of N superfields Φ^i. The individual particle scattering amplitudes $A_N^{(\text{tree})}$ are then be extracted as coefficients of an expansion in the Graßmann parameters following (1.20). The classification of amplitudes in terms of helicity violation carries over to the superamplitudes. To see this, we expand the superamplitude as

$$\begin{array}{ccccccccccc} \mathcal{A}_N^{(\text{tree})} = & \mathcal{A}_{N,2}^{(\text{tree})} & + & \mathcal{A}_{N,3}^{(\text{tree})} & + & \mathcal{A}_{N,4}^{(\text{tree})} & + \ldots + & \mathcal{A}_{N,N-3}^{(\text{tree})} & + & \mathcal{A}_{N,N-2}^{(\text{tree})} & , \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \text{MHV} & & \text{NMHV} & & \text{N}^2\text{MHV} & & \overline{\text{NMHV}} & & \overline{\text{MHV}} & \end{array} \tag{1.22}$$

where $\mathcal{A}_{N,K}^{(\text{tree})}$ contains products of $4K$ Graßmann variables $\eta^i_{\dot{a}}$. Therefore the helicity $H = \sum_i h^i = N - 2K$ of all particle amplitudes packaged in the superamplitude $\mathcal{A}_{N,K}^{(\text{tree})}$ is identical. In particular, one expansion coefficient of $\mathcal{A}_{N,K}^{(\text{tree})}$ is identical to the gluon amplitude $A_{N,K}^{(\text{tree})}$ with helicity H. Hence the classification of gluon amplitudes in terms of the degree of helicity violation can be extended to the superamplitudes as indicated in (1.22). See once again figure 1.1.

After setting up the formalism we can discuss actual superamplitudes. We confine ourselves to the supersymmetric generalizations of the MHV gluon amplitudes $A_{N,2}^{(\text{tree})}$ and the NMHV amplitude $A_{6,3}^{(\text{tree})}$, which also served as illustrative examples in the previous section. The MHV superamplitude is [77]

$$\mathcal{A}_{N,2}^{(\text{tree})} = \frac{\delta^{4|0}(P)\delta^{0|8}(Q)}{\langle 12\rangle\langle 23\rangle \cdots \langle N-1\, N\rangle\langle N1\rangle} \tag{1.23}$$

for $N \geq 4$. The momentum conserving delta function $\delta^{4|0}(P) \equiv \delta^4(P)$ is given by (1.16), where the notation here stresses that it is purely bosonic. The other delta function implements what is called supermomentum conservation,

$$\delta^{0|8}(Q) = \prod_{\alpha=1}^{2}\prod_{\dot{a}=1}^{4} Q_{\alpha\dot{a}} \quad \text{with} \quad Q_{\alpha\dot{a}} = \sum_{i=1}^{N} q^i_{\alpha\dot{a}} \quad \text{and} \quad q^i_{\alpha\dot{a}} = \lambda^i_\alpha \eta^i_{\dot{a}}\,, \tag{1.24}$$

where in the product of anticommuting factors those with smaller indices appear left. Let us discuss in more detail how to extract particle amplitudes from this quantity. For this purpose we display some parts of the Graßmann expansion of (1.23) explicitly,

$$\begin{aligned} \mathcal{A}_{N,2}^{(\text{tree})} = \quad & \ldots + (\eta^i_1\eta^i_2\eta^i_3\eta^i_4)(\eta^j_1\eta^j_2\eta^j_3\eta^j_4) \\ & \qquad \cdot A_{N,2}^{(\text{tree})}(g^1_+, \ldots, g^i_-, \ldots, g^j_-, \ldots, g^N_+) \\ & + \ldots + (\eta^k_1\eta^k_2\eta^k_3\eta^k_4)(\eta^l_2)(-\eta^m_2\eta^m_3\eta^m_4) \\ & \qquad \cdot A_N^{(\text{tree})}(g^1_+, \ldots, g^k_-, \ldots, \tilde{g}^l_2, \ldots, \bar{\tilde{g}}^m_1, \ldots, g^N_+) \\ & + \ldots \,. \end{aligned} \tag{1.25}$$

Comparing with the Graßmann expansion of the superfield (1.20), we identify the MHV gluon amplitude (1.15). As an illustration we displayed a further term. According to (1.20) this has to be an amplitude involving gluons $g_\pm$ as well as a gluino $\tilde{g}_2$ and an antigluino $\bar{\tilde{g}}_1$. Adding up the helicities of these particles we obtain the total helicity $H = \sum_i h^i = N - 4$, which is the same as that of the MHV gluon amplitude. Hence these amplitudes are contained in the same superamplitude $\mathcal{A}_{N,2}^{(\text{tree})}$.

We move on to the six-particle NMHV superamplitude [78, 79]

$$\mathcal{A}_{6,3}^{(\text{tree})} = \frac{\delta^{4|0}(P)\delta^{0|8}(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\left(\mathcal{R}^{1;46} + \mathcal{R}^{1;35} + \mathcal{R}^{1;36}\right). \tag{1.26}$$

In order to define the quantities $\mathcal{R}^{r;st}$ in this formula, we introduce so-called *dual variables* $x^i_{\alpha\dot\beta}$ and $\theta^i_{\alpha\dot b}$ by the relations

$$p^i_{\alpha\dot\beta} = \lambda^i_\alpha\tilde\lambda^i_{\dot\beta} = x^i_{\alpha\dot\beta} - x^{i+1}_{\alpha\dot\beta}\,, \quad q^i_{\alpha\dot a} = \lambda^i_\alpha\eta^i_{\dot a} = \theta^i_{\alpha\dot a} - \theta^{i+1}_{\alpha\dot a}\,, \tag{1.27}$$

where $x^{N+1} = x^1$ and $\theta^{N+1} = \theta^1$. Furthermore, we define the abbreviations $x^{ij} = x^i - x^j$ and $\theta^{ij} = \theta^i - \theta^j$. Then

$$\mathcal{R}^{r;st} = \frac{\langle s\, s-1\rangle\langle t\, t-1\rangle\delta^{0|4}\big(\langle r|x^{rs}x^{st}|\theta^{tr}\rangle + \langle r|x^{rt}x^{ts}|\theta^{sr}\rangle\big)}{(x^{st})^2\langle r|x^{rs}x^{st}|t\rangle\langle r|x^{rs}x^{st}|t-1\rangle\langle r|x^{rt}x^{ts}|s\rangle\langle r|x^{rt}x^{ts}|s-1\rangle}\,, \tag{1.28}$$

where the fermionic delta function is defined analogous to the one in (1.24). The brackets occurring in this expression can be reduced to ordinary spinor brackets, which we illustrate for one example: $\langle 6|x^{64}x^{42}|\theta^{26}\rangle = \langle 6(\,-4\rangle[4-5\rangle[5)(\,-2]\langle 2-3]\langle 3)(\,-1\rangle\eta^1 - 2\rangle\eta^2)$. The gluon NMHV amplitude (1.17) can be recovered from (1.26) by means of an expansion in the Graßmann parameters. We picked a certain order of gluon helicities in (1.17) to obtain a particularly simple formula. The superamplitude in (1.26) contains the gluon amplitudes for any order of helicities. Let us also comment on the dual variables x^i and θ^i introduced in (1.27). As we will discuss in the subsequent section 1.3.4, amplitudes have a superconformal symmetry as functions of the momenta p^i and the supermomenta q^i. Remarkably, they exhibit a second so-called *dual superconformal symmetry* in the variables x^i and θ^i. The quantities $\mathcal{R}^{r;st}$ are superconformal as well as dual superconformal invariants and they appear naturally as residues of certain integrals that we will introduce in section 1.3.5.

All amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ in $\mathcal{N} = 4$ SYM were determined in [80] by solving a supersymmetric generalization of the BCFW recursion relations [81, 82, 83]. As in the bosonic version, these relations make use of complexified particle momenta. The starting point of the recursion are the three-particle scattering amplitudes[6]

$$\begin{aligned}\mathcal{A}_{3,2}^{(\text{tree})} &= \frac{\delta^{4|0}(P)\delta^{0|8}(Q)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,,\\ \mathcal{A}_{3,1}^{(\text{tree})} &= \frac{\delta^{4|0}(P)\delta^{0|4}([23]\eta^1 + [31]\eta^2 + [12]\eta^3)}{[23][31][12]}\end{aligned} \tag{1.29}$$

from which higher point amplitudes are constructed. For three real null vectors, momentum conservation immediately implies that all spinor brackets $\langle ij\rangle$ and $[ij]$ vanish. Hence, the three-particle amplitudes are only meaningful in a complexified setting where the spinors λ and $\tilde\lambda$ are treated as independent complex variables not obeying (1.5). We present these amplitudes at this point because they will reappear prominently in sections 1.3.5 and 1.3.6.

[6]The amplitude $\mathcal{A}_{3,2}^{(\text{tree})}$ can be obtained from the formula (1.23) for MHV superamplitudes for $N = 3$.

1.3.4 Symmetries and Integrability

After introducing the tree-level partial gluon amplitudes $A_{N,K}^{(\text{tree})}$ and their supersymmetric generalizations $\mathcal{A}_{N,K}^{(\text{tree})}$ in the previous sections, we move on to present their most important properties. From our perspective, these are their symmetries because they severely constrain and possibly even completely determine the amplitudes. We will discuss a finite-dimensional Lie (super)algebra symmetry and a certain infinite-dimensional extension thereof called *Yangian* symmetry [84]. This Yangian symmetry algebra is closely tied to integrability. Important integrable spin chain models, such as the famous Heisenberg ferromagnet, and also integrable $1+1$-dimensional quantum field theories are governed by it, see e.g. [18, 19] and section 1.1. Furthermore, the Yangian of the superconformal algebra $\mathfrak{psu}(2,2|4)$ underlies the one-loop spin chain of the planar $\mathcal{N}=4$ SYM spectral problem [39, 40], cf. section 1.2. The very same Yangian algebra was also found in the study of the scattering amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ of that theory [52], see [57, 85, 86] for reviews. This discovery strengthened the believe that integrability is not just a feature of the spectral problem but controls all observables in planar $\mathcal{N}=4$ SYM.

Before becoming more technical, let us briefly sketch how the Yangian symmetry of scattering amplitudes was discovered. The foundation is a well established Lie (super)algebra symmetry. The gluon amplitudes $A_{N,K}^{(\text{tree})}$ are annihilated by all generators of the conformal algebra $\mathfrak{su}(2,2)$ acting on the particle momenta p^i. Analogously, the superamplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ are annihilated by the generators of the superconformal algebra $\mathfrak{psu}(2,2|4)$ that act on p^i and on the supermomenta q^i. An illustrative calculation showing this for the MHV superamplitudes is contained in [67]. Besides, a second copy of $\mathfrak{psu}(2,2|4)$ termed *dual superconformal symmetry* that acts on the dual variables x^i and θ^i, cf. (1.27), was found [87, 78, 81]. Combining these two superconformal algebras leads to the Yangian of $\mathfrak{psu}(2,2|4)$ and consequently to the *Yangian invariance* of the superamplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ [52].

We continue by defining the Yangian invariance of amplitudes on a technical level. To this end, we deviate from the "historic" route of introducing the dual superconformal symmetry as just outlined. We first discuss the Lie superalgebra symmetry, which will be extended to a Yangian symmetry later. As $\mathcal{A}_{N,K}^{(\text{tree})}$ is annihilated by all generators of the ordinary superconformal algebra $\mathfrak{psu}(2,2|4)$, it is also annihilated by complex linear combinations thereof and thus by the complexified algebra $\mathfrak{psl}(\mathbb{C}^{4|4}) \equiv \mathfrak{psl}(4|4)$. Hence, we can work with complex algebras. Our aim is to state the generators that annihilate the superamplitude. For this purpose, we start with a set of generators $\mathfrak{J}_{\mathcal{A}\mathcal{B}}$ of the superalgebra $\mathfrak{gl}(4|4) \supset \mathfrak{psl}(4|4)$, which are easily realized in terms of spinor helicity variables. Arranged into a supermatrix they read

$$
(\mathfrak{J}_{\mathcal{A}\mathcal{B}}) = \left(\begin{array}{c} \lambda_\alpha \\ \hline -\partial_{\tilde\lambda_{\dot\alpha}} \\ \hline \partial_{\eta_{\dot a}} \end{array}\right) \left(\begin{array}{c|c|c} \partial_{\lambda_\beta} & \tilde\lambda_{\dot\beta} & \eta_{\dot b} \end{array}\right) = \left(\begin{array}{c|c|c} \lambda_\alpha\partial_{\lambda_\beta} & \lambda_\alpha\tilde\lambda_{\dot\beta} & \lambda_\alpha\eta_{\dot b} \\ \hline -\partial_{\tilde\lambda_{\dot\alpha}}\partial_{\lambda_\beta} & -\partial_{\tilde\lambda_{\dot\alpha}}\tilde\lambda_{\dot\beta} & -\partial_{\tilde\lambda_{\dot\alpha}}\eta_{\dot b} \\ \hline \partial_{\eta_{\dot a}}\partial_{\lambda_\beta} & \partial_{\eta_{\dot a}}\tilde\lambda_{\dot\beta} & \partial_{\eta_{\dot a}}\eta_{\dot b} \end{array}\right). \tag{1.30}
$$

Here we split the superindex $\mathcal{A}=1,\ldots,8$ into bosonic indices $\alpha,\dot\alpha=1,2$ with degree $|\alpha|=|\dot\alpha|=0$ and a fermionic index $\dot a=1,2,3,4$ with degree $|\dot a|=1$. The generators obey the commutation relations

$$
[\mathfrak{J}_{\mathcal{A}\mathcal{B}},\mathfrak{J}_{\mathcal{C}\mathcal{D}}\} = \delta_{\mathcal{C}\mathcal{B}}\mathfrak{J}_{\mathcal{A}\mathcal{D}} - (-1)^{(|\mathcal{A}|+|\mathcal{B}|)(|\mathcal{C}|+|\mathcal{D}|)}\delta_{\mathcal{A}\mathcal{D}}\mathfrak{J}_{\mathcal{C}\mathcal{B}}\,, \tag{1.31}
$$

where the left hand side denotes the graded commutator, see section 2.1 for details on superalgebras. To obtain the algebra $\mathfrak{psl}(4|4)$ we have to study the center of $\mathfrak{gl}(4|4)$, i.e.

those elements whose graded commutator with all others vanishes. It is spanned by two generators,

$$\mathfrak{C} = \mathrm{tr}(\mathfrak{J}_{\mathcal{AB}}) = \sum_{\mathcal{A}} \mathfrak{J}_{\mathcal{AA}} = \sum_{\alpha=1}^{2} \lambda_\alpha \partial_{\lambda_\alpha} - \sum_{\dot\alpha=1}^{2} \tilde\lambda_{\dot\alpha} \partial_{\tilde\lambda_{\dot\alpha}} - \sum_{\dot a=1}^{4} \eta_{\dot a} \partial_{\eta_{\dot a}} + 2 \tag{1.32}$$

and $\mathfrak{B} = \mathrm{str}(\mathfrak{J}_{\mathcal{AB}}) = \sum_{\mathcal{A}} (-1)^{|\mathcal{A}|} \mathfrak{J}_{\mathcal{AA}}$. The subalgebra $\mathfrak{sl}(4|4) \subset \mathfrak{gl}(4|4)$ is defined by imposing the relation $\mathfrak{B} = 0$. A set of $\mathfrak{sl}(4|4)$ generators is

$$\mathfrak{J}'_{\mathcal{AB}} = \mathfrak{J}_{\mathcal{AB}} - \frac{1}{8}(-1)^{|A|} \delta_{\mathcal{AB}} \mathfrak{B} \tag{1.33}$$

satisfying $\mathfrak{B}' = 0$ and $\mathfrak{C}' = \mathfrak{C}$. One obtains the simple algebra $\mathfrak{psl}(4|4) \subset \mathfrak{sl}(4|4)$ by demanding that also $\mathfrak{C} = 0$. Generators $\mathfrak{J}^i_{\mathcal{AB}}$ of the form (1.30) act on each particle of the superamplitude $\mathcal{A}^{(\mathrm{tree})}_{N,K}$. Thus an action of $\mathfrak{gl}(4|4)$ on the whole amplitude is

$$M^{[1]}_{\mathcal{AB}} = \sum_{i=1}^{N} \mathfrak{J}^i_{\mathcal{BA}} \,. \tag{1.34}$$

We already argued that superamplitudes are invariant under $\mathfrak{psl}(4|4)$. Noting that $\mathfrak{C}^i = 0$ in (1.32) agrees with the condition on the superhelicity in (1.21), the superamplitudes are even invariant under $\mathfrak{sl}(4|4)$,

$$M'^{[1]}_{\mathcal{AB}} \mathcal{A}^{(\mathrm{tree})}_{N,K} = 0 \,, \tag{1.35}$$

where the prime indicates that the generators $\mathfrak{J}^i_{\mathcal{AB}}$ are to be replaced by $\mathfrak{J}'^i_{\mathcal{AB}}$. This concludes the discussion of the Lie superalgebra invariance.

The Yangian extension of $\mathfrak{gl}(4|4)$ is obtained by appending to the Lie superalgebra generators $M^{[1]}_{\mathcal{AB}}$ an infinite set of further generators $M^{[l]}_{\mathcal{AB}}$ indexed by an integer $l > 1$. Often the generators $M^{[l]}_{\mathcal{AB}}$ are said to be of level $l-1$. There exists an elegant construction of these generators within the so-called quantum inverse scattering framework, which we will elaborate on in section 2.1. At this point, however, we choose a rather pedestrian approach and just state the explicit form of the generators with $l = 2$,

$$\begin{aligned} M^{[2]}_{\mathcal{AB}} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{N} \sum_{\mathcal{C}} (-1)^{|\mathcal{C}|} \Big(\mathfrak{J}^j_{\mathcal{BC}} \mathfrak{J}^i_{\mathcal{CA}} - \mathfrak{J}^i_{\mathcal{BC}} \mathfrak{J}^j_{\mathcal{CA}} \Big) \\ + \sum_{i=1}^{N} \Big(v_i \mathfrak{J}^i_{\mathcal{BA}} - \frac{1}{2} \sum_{\mathcal{C}} (-1)^{|\mathcal{A}||\mathcal{B}|+|\mathcal{A}||\mathcal{C}|+|\mathcal{B}||\mathcal{C}|} \mathfrak{J}^i_{\mathcal{CA}} \mathfrak{J}^i_{\mathcal{BC}} \Big) \,, \end{aligned} \tag{1.36}$$

where v_i are arbitrary complex parameters. This suffices because all generators with larger l can constructed from these. The graded commutator between generators with $l = 1, 2$ evaluates to

$$[M^{[1]}_{\mathcal{AB}}, M^{[2]}_{\mathcal{CD}}\} = \delta_{\mathcal{AD}} M^{[2]}_{\mathcal{CB}} - (-1)^{(|\mathcal{A}|+|\mathcal{B}|)(|\mathcal{C}|+|\mathcal{D}|)} \delta_{\mathcal{CB}} M^{[2]}_{\mathcal{AD}} \,. \tag{1.37}$$

We move on to the application of this Yangian algebra to scattering amplitudes. For this we need a special case of the generators (1.36). Form (1.30) one derives

$$\sum_{\mathcal{C}} (-1)^{|\mathcal{A}||\mathcal{B}|+|\mathcal{A}||\mathcal{C}|+|\mathcal{B}||\mathcal{C}|} \mathfrak{J}^i_{\mathcal{CA}} \mathfrak{J}^i_{\mathcal{BC}} = (\mathfrak{C}^i - 1) \mathfrak{J}^i_{\mathcal{BA}} + (-1)^{|\mathcal{A}|} \delta_{\mathcal{AB}} \mathfrak{C}^i \,. \tag{1.38}$$

Using this identity, the fact that $\mathfrak{C}^i = 0$ on the amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ and setting $v_i = -\frac{1}{2}$, we rewrite (1.36) as

$$M_{\mathcal{AB}}^{[2]} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{N} \sum_{\mathcal{C}} (-1)^{|\mathcal{C}|} \left(\mathfrak{J}_{\mathcal{BC}}^{j} \mathfrak{J}_{\mathcal{CA}}^{i} - \mathfrak{J}_{\mathcal{BC}}^{i} \mathfrak{J}_{\mathcal{CA}}^{j} \right) . \tag{1.39}$$

This is form of the Yangian generators is often found in the literature on scattering amplitudes, cf. [52]. The superamplitudes are annihilated by these generators,

$$M_{AB}'^{[2]} \mathcal{A}_{N,K}^{(\text{tree})} = 0 \,, \tag{1.40}$$

where again the prime indicates that $\mathfrak{gl}(4|4)$ generators $\mathfrak{J}_{\mathcal{AB}}^{i}$ are replaced by the $\mathfrak{sl}(4|4)$ generators $\mathfrak{J}_{\mathcal{AB}}'^{i}$.[7] The prove of this condition constitutes the main result of [52]. Equations (1.35) and (1.40) combined imply the invariance of the superamplitudes under the Yangian of $\mathfrak{sl}(4|4)$. The exploration of Yangian invariants of this and other algebras will be the main subject of this thesis.

We finish this section with some remarks. Apart from the continuous symmetries addressed here, amplitudes have a number of discrete symmetries. Some of these follow directly from the definition of the partial amplitudes [58]. As one example let us mention the invariance of $\mathcal{A}_{N,K}^{(\text{tree})}$ under a cyclic shift of the particles $i \mapsto i+1$. This cyclic invariance can also be seen in the Yangian generators [52] when acting on an amplitude, manifestly in (1.34) and non-manifestly in (1.36). Further relations between the $(N-1)!$ partial amplitudes, that enter the full scattering amplitude in (1.1), reduce the number of independent ones to $(N-3)!$ as shown in [88].

Another remark concerns the superconformal (1.35) and Yangian invariance (1.40) of the amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$. A careful analysis taking into account the reality conditions (1.5) of the spinor variables reveals that these symmetries can be violated if two particle momenta become collinear and thus a spinor bracket in the denominator of the amplitude vanishes. This phenomenon can in principle be overcome by introducing correction terms to the generators of the symmetry algebra [89]. Furthermore, the symmetries may break down at multi-particle poles of the amplitude [90]. All of these issues are reviewed in [91]. Because they do not occur for generic particle momenta, they are often neglected in the discussion of tree-level amplitudes. This is also how we proceed in most of this thesis.

It is worth emphasizing that although the amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ are annihilated by the Yangian of the complex algebra $\mathfrak{sl}(4|4)$, the real form $\mathfrak{su}(2,2|4)$ is of importance in the discussion of scattering amplitudes. As a drastic illustration let us mention that below in sections 2.4.2.3 and 4.1.5.2 we will construct invariants associated with the compact real form $\mathfrak{su}(4|4)$. Also these invariants are annihilated by the Yangian of $\mathfrak{sl}(4|4)$, however they are just polynomials and not distributions like the amplitudes.

1.3.5 Graßmannian Integral

So far we presented explicit formulas only for a few superamplitudes. In (1.23) we introduced the MHV superamplitudes $\mathcal{A}_{N,2}^{(\text{tree})}$ and in (1.26) we displayed the simplest NMHV superamplitude $\mathcal{A}_{6,3}^{(\text{tree})}$. Comparing these two equations, we notice that the complexity of the expression increases significantly from MHV to NMHV superamplitudes.

[7] A short calculation shows that the Yangian generators in (1.39) are not affected by this replacement, $M_{\mathcal{AB}}'^{[2]} = M_{\mathcal{AB}}^{[2]}$.

As already mentioned in the context of these equations, an explicit formula for all $\mathcal{A}_{N,K}^{(\text{tree})}$ is known [80]. However, for an decreasing degree of helicity violation the representation of the superamplitude that this formula produces gets more and more involved. In this section we discuss an alternative and very compact formulation of these superamplitudes in terms of certain multi-dimensional contour integrals, so-called *Graßmannian integrals* [92], see also [93]. The matrix-valued integration variable can be interpreted as a point in a Graßmannian manifold, hence the name. This approach to superamplitudes especially suites our interests because the compact Graßmannian integral formula allows for an easy investigation of symmetries. While the superconformal symmetry is manifest in this formula, also the Yangian symmetry can be verified [94, 95].

Before defining the Graßmannian integral, we first have to discuss some bare essentials about Graßmannian manifolds. Many more details on this topic can be found in books on algebraic geometry like e.g. [96]. The *Graßmannian* $\mathrm{Gr}(N, K)$ is the space of all K-dimensional linear subspaces of $\mathbb{C}^N$. The complex entries of a $K \times N$ matrix C provide "homogeneous" coordinates on this space. The transformation $C \mapsto VC$ with $V \in GL(\mathbb{C}^K)$ corresponds to a change of basis within a given subspace, and thus it does not change the point in the Graßmannian. This allows us to describe a generic point in $\mathrm{Gr}(N, K)$ by the "gauge fixed" matrix

$$C = \left(\begin{array}{c:c} 1_K & \mathcal{C} \end{array} \right) \quad \text{with} \quad \mathcal{C} = \begin{pmatrix} C_{1K+1} & \cdots & C_{1N} \\ \vdots & & \vdots \\ C_{KK+1} & \cdots & C_{KN} \end{pmatrix}, \tag{1.41}$$

where 1_K denotes the $K \times K$ unit matrix. In what follows we will also encounter the $(N-K) \times N$ matrix

$$C^\perp = \left(\begin{array}{c:c} -\mathcal{C}^t & 1_{N-K} \end{array} \right) \tag{1.42}$$

that obeys $C(C^\perp)^t = 0$. It may be considered as an element of $\mathrm{Gr}(N, N-K)$. These ingredients are sufficient to present the *Graßmannian integral* formulation of $\mathcal{N} = 4$ SYM superamplitudes [92],

$$\mathcal{A}_{N,K}^{(\text{tree})} = \int \mathrm{d}\mathcal{C} \frac{\delta_*^{2(N-K)|0}(C^\perp \boldsymbol{\lambda}) \delta_*^{2K|0}(C\tilde{\boldsymbol{\lambda}}) \delta^{0|4K}(C\boldsymbol{\eta})}{(1, \ldots, K) \cdots (N, \ldots, K-1)} \tag{1.43}$$

with the holomorphic $K(N-K)$-form $\mathrm{d}\mathcal{C} = \bigwedge_{k,l} \mathrm{d}C_{kl}$. In this formula $(i, \ldots, i+K-1)$ denotes the minor of the matrix C consisting of the consecutive columns $i, \ldots, i+K-1$. These are counted modulo N such that they are in the range $1, \ldots, N$. The external data is encoded in the $N \times 2$ matrices $\boldsymbol{\lambda} = (\lambda_\alpha^i)$ and $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_{\dot\alpha}^i)$ as well as the $N \times 4$ matrix $\boldsymbol{\eta} = (\eta_{\dot a}^i)$. The symbol δ_* denotes a formal bosonic delta function whose argument may be complex. It can be understood as a calculation rule to set the argument to zero.

Let us interject that in this thesis we encounter different types of bosonic delta functions besides δ_*. An ordinary delta function of a real argument is always simply denoted by δ, see e.g. (1.16). Below in chapter 4 we will encounter a complex delta function $\delta_{\mathbb{C}}$ which is defined in terms of real ones.

We now discuss the evaluation of the Graßmannian integral (1.43). For this we have to specify the contour of integration. Before doing so, it is helpful to note that the equations imposed by the bosonic and fermionic delta functions in (1.43) imply momentum conservation (1.16) and supermomentum conservation (1.24),

$$\begin{aligned} \boldsymbol{\lambda}^t \tilde{\boldsymbol{\lambda}} = 0 \quad &\Leftrightarrow \quad P_{\alpha\dot\beta} = 0\,, \\ \boldsymbol{\lambda}^t \boldsymbol{\eta} = 0 \quad &\Leftrightarrow \quad Q_{\alpha\dot b} = 0\,. \end{aligned} \tag{1.44}$$

Of course, these constraints are understood to hold only in the presence of those delta function. This allows us to assess the number of integration variables remaining of the $K(N-K)$ variables contained in $\mathcal{C}$ after solving for the $2N$ bosonic delta functions. Taking into account that due to momentum conservation there are four bosonic delta functions remaining, we are left with

$$K(N-K)-2N+4 \tag{1.45}$$

complex integration variables. Thus a contour has to be specified only for these variables. Let us first focus on those sample amplitudes with which we are already familiar with from the previous sections. For the MHV amplitudes $\mathcal{A}_{N,2}^{(\text{tree})}$ there is no integration remaining according to (1.45). Thus the Graßmannian integral (1.43) directly evaluates to the Parke-Taylor-like formula (1.23) after solving the bosonic delta functions. The next example is once again the NMHV amplitude $\mathcal{A}_{6,3}^{(\text{tree})}$, where we are left with one complex integration variable. The integrand has six poles in this variable, that are related to the points where the minors in (1.43) vanish. The expression (1.26) for $\mathcal{A}_{6,3}^{(\text{tree})}$ is then obtained by picking a closed contour which encircles three of those poles and applying Cauchy's residue theorem. Each of the three terms in (1.26) corresponds to one residue. At this point we can illustrate the situation for a general amplitude $\mathcal{A}_{N,K}^{(\text{tree})}$. According to (1.45) we are left with a multi-dimensional complex contour integral. In order to obtain a Yangian invariant expression one should select a closed contour because the integrand of (1.43) is only Yangian invariant up to an exact term [94, 95]. This type of integrals can be evaluated by means of a multi-dimensional generalization of Cauchy's residue theorem, the so-called "global residue theorem", see the discussion in [92] and the references given there. An explicit contour for all amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ was proposed in [97], see also [98, 99].

Let us also mention some open challenges of the Graßmannian integral approach in the form presented in (1.43). The first point concerns the spacetime signature. In case of the physical Minkowski signature $(1,3)$, the spinors obey the reality conditions (1.5). The spinors contained in $\tilde{\boldsymbol{\lambda}}$ depend on those in $\boldsymbol{\lambda}$ and both variables are in general complex. Hence, the bosonic delta functions δ_* in (1.43) have complex arguments and therefore can strictly speaking not be treated as ordinary real delta functions δ. Moreover, the counting used to obtain the number of remaining integrations in (1.45) is invalid because it assumes the independence of $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$. Typically these issues are avoided by either working in split signature $(2,2)$ or in a complexified momentum space, where $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$ are treated as independent real or complex variables, respectively. In the arguments presented in this section we implicitly worked with the latter choice. The second point to be discussed is the contour of integration. As just mentioned, a contour for all amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ was given in [97]. However, the explicit form of this contour is quite intricate. One might argue that the complexity of explicit formulas for general amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$, which served as a motivation for the Graßmannian integral formulation in the first paragraph, actually persists in this approach. While the integrand of (1.43) is simple, the complexity is contained in the contour. In this thesis we will address both points. We will argue that in Minkowski signature the reality conditions of the spinors and the choice of the integration contour are tightly interrelated.

The Graßmannian integral formulation of scattering amplitudes as introduced in [92] presents merely the initial step in a plethora of in part still ongoing developments. Let us briefly mention some of the major advances of these investigations. While in this section we confined ourselves to the Graßmannian integral (1.43) for tree-level superamplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$, this integral also contains certain data of loop amplitudes, cf. (1.2). Already in [92] it was argued that with a suitable contour the integral computes leading singularities of loop amplitudes. Further major steps were performed in [100]. In this work tree-level amplitudes and even all-loop integrands were formulated in terms of *on-shell diagrams*. These are networks composed out of the two types of three-particle amplitudes in (1.29), which are represented graphically by trivalent vertices. It was realized that the on-shell diagrams can be described utilizing the geometry of Graßmannian manifolds and each diagram can be labeled by a permutation. The Graßmannian integral of [92] is understood in this setting as a means of computing on-shell diagrams. To obtain amplitudes a number of on-shell diagrams have to be added up. Another development is the introduction of a structure named *Amplituhedron* [101], which aims at interpreting amplitudes in geometric terms as certain "volumes". Finally, we want to emphasize that while the material presented in the main part of this thesis has certainly interesting connections to most of the developments sketched here, we will mostly confine ourselves to relating it to the original Graßmannian integral as introduced in [92] and presented around (1.43).

1.3.6 Integrable Deformations

We already encountered one connection between scattering amplitudes and integrability in section 1.3.4, namely the Yangian invariance discovered in [52]. A different link was observed in [102], where in particular the amplitude $\mathcal{A}_{4,2}^{(\text{tree})}$ was related to the one-loop dilatation operator of the planar $\mathcal{N}=4$ SYM spectral problem. This operator is the Hamiltonian of an integrable spin chain and it can be constructed employing an R-matrix, which is a solution of the Yang-Baxter equation [39, 40]. This R-matrix depends on an arbitrary free complex parameter, a so-called spectral parameter. The authors of [103, 104] set out to expose this parameter also in the context of scattering amplitudes, thus leading to *deformed amplitudes*. The reasons for pursuing this route are manifold. First, these deformations are crucial to identify structures that are necessary to apply powerful integrability-based methods like the quantum inverse scattering framework to the study of amplitudes. In analogy to the situation for the spectral problem almost 15 years ago, such an understanding of tree-level amplitudes should also provide important clues of how to proceed to loop-level. Second, as put forward in [103, 104], the deformation parameters could even be of direct relevance for loop amplitudes as novel symmetry preserving regulators of divergent loop integrals. Finally, the deformations are of interest because they reveal exciting connections between amplitudes and various areas of mathematics, ranging from representation theory to hypergeometric functions. In the main part of this thesis we will detail this enumeration and add some further items. Let us mention that the developments reviewed in this section happened in parallel to the author's research presented later in this thesis.

The key principle in constructing deformed amplitudes $\mathcal{A}_{N,K}^{(\text{def.})}$ is that they remain Yangian invariant. We already know that the Yangian generators $M_{\mathcal{AB}}^{[2]}$ in (1.39), which appear for the undeformed amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$, are a very particular case of (1.36). The latter equation can be rephrased as

$$M_{\mathcal{AB}}^{[2]}=\frac{1}{2}\sum_{\substack{i,j=1\\i<j}}^{N}\sum_{\mathcal{C}}(-1)^{|\mathcal{C}|}\Big(\mathfrak{J}_{\mathcal{BC}}^{j}\mathfrak{J}_{\mathcal{CA}}^{i}-\mathfrak{J}_{\mathcal{BC}}^{i}\mathfrak{J}_{\mathcal{CA}}^{j}\Big)+\sum_{i=1}^{N}\hat{v}_i\mathfrak{J}_{\mathcal{BA}}^{i}\,, \tag{1.46}$$

where we used (1.38), assumed $\sum_{i=1}^{N}\mathfrak{C}^i=0$ and introduced $\hat{v}_i=v_i-\frac{c_i}{2}+\frac{1}{2}$ with c_i being the eigenvalue of $\mathfrak{C}^i$. Deformed amplitudes are then characterized by the Yangian invariance condition

$$M'^{[1]}_{\mathcal{AB}}\mathcal{A}^{(\text{def.})}_{N,K} = 0\,, \quad M'^{[2]}_{\mathcal{AB}}\mathcal{A}^{(\text{def.})}_{N,K} = 0\,, \tag{1.47}$$

where $M^{[1]}_{\mathcal{AB}}$ is still given by (1.34) and $M^{[2]}_{\mathcal{AB}}$ by (1.46). Furthermore, the prime signifies that the $\mathfrak{gl}(4|4)$ generators $\mathfrak{J}^i_{\mathcal{AB}}$ are to be replaced by the $\mathfrak{sl}(4|4)$ generators $\mathfrak{J}'^i_{\mathcal{AB}}$ in (1.33). These deformations comprise the complex parameters $\hat{v}_i$ and c_i. The parameters $\hat{v}_i$ appear directly in the Yangian generators (1.46). The solutions $\mathcal{A}^{(\text{def.})}_{N,K}$ to the Yangian invariance condition also contain the c_i, which can be understood as deformations of the superhelicities, cf. (1.21) and (1.32).

As a first example we state the deformation of the four-particle MHV amplitude [103, 104],

$$\mathcal{A}^{(\text{def.})}_{4,2} = \frac{\delta^{4|0}(P)\delta^{0|8}(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(\frac{\langle 41\rangle}{\langle 34\rangle}\right)^{c_1}\left(\frac{\langle 12\rangle}{\langle 41\rangle}\right)^{c_2}\left(\frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 41\rangle}\right)^{z} \tag{1.48}$$

with $z = \hat{v}_1 + \frac{c_1}{2} - \hat{v}_2 - \frac{c_2}{2}$. The eigenvalues of the central elements and the parameters in the Yangian generators obey

$$c_3 = -c_1\,, \quad c_4 = -c_2\,, \quad \hat{v}_3 = \hat{v}_1\,, \quad \hat{v}_4 = \hat{v}_2\,. \tag{1.49}$$

The Yangian invariance condition (1.47) for this deformed amplitude can be shown to be equivalent to the Yang-Baxter equation. Hence, $\mathcal{A}^{(\text{def.})}_{4,2}$ is essentially an R-matrix with spectral parameter z and c_1, c_2 may be interpreted as representation labels. In this language the undeformed amplitude $\mathcal{A}^{(\text{tree})}_{4,2}$ is understood as an R-matrix with the representations $c_1 = c_2 = 0$ that is evaluated at a special value $z = 0$ of the spectral parameter. Let us interject that this interpretation proposed in [103, 104] differs slightly from the one in [102], where the amplitude is related to a Hamiltonian and not to an R-matrix. Note that typically such a Hamiltonian of an integrable spin chain is not Yangian invariant, see the discussion at the end of section 2.2 below. We believe that the conceptual differences between the two interpretations deserve further attention. However, in this thesis we do not dwell on this point and stick to that of [103, 104].

We move on to the deformations of the two complexified three-particle amplitudes (1.29) that were also introduced in [103, 104]. The first one reads

$$\mathcal{A}^{(\text{def.})}_{3,2} = \frac{\delta^{4|0}(P)\delta^{0|8}(Q)}{\langle 12\rangle^{1+c_3}\langle 23\rangle^{1+c_1}\langle 31\rangle^{1+c_2}} \tag{1.50}$$

with

$$\hat{v}_3 - \frac{c_3}{2} = \hat{v}_1 + \frac{c_1}{2}\,, \quad \hat{v}_1 - \frac{c_1}{2} = \hat{v}_2 + \frac{c_2}{2}\,, \quad \hat{v}_2 - \frac{c_2}{2} = \hat{v}_3 + \frac{c_3}{2}\,. \tag{1.51}$$

The second one becomes

$$\mathcal{A}^{(\text{def.})}_{3,1} = \frac{\delta^{4|0}(P)\delta^{0|4}([23]\eta^1 + [31]\eta^2 + [12]\eta^3)}{[23]^{1+c_1}[31]^{1+c_2}[12]^{1+c_3}} \tag{1.52}$$

with

$$\hat{v}_2 - \frac{c_2}{2} = \hat{v}_1 + \frac{c_1}{2}\,, \quad \hat{v}_3 - \frac{c_3}{2} = \hat{v}_2 + \frac{c_2}{2}\,, \quad \hat{v}_1 - \frac{c_1}{2} = \hat{v}_3 + \frac{c_3}{2}\,. \tag{1.53}$$

For these deformed three-particle amplitudes the Yangian invariance condition (1.47) turns out to be equivalent to a so-called bootstrap equation. This equation is known from the description of bound states in $1+1$-dimensional integrable models, see [105] and e.g. [106, 107].

Having these deformed three-point amplitudes at hand, one can start deforming the on-shell diagrams of [100], which are constructed by gluing together these amplitudes. In particular, the deformation of an on-shell diagram for the one-loop amplitude $\mathcal{A}^{(1)}_{4,2}$ was investigated in [103, 104]. In the undeformed case this on-shell diagram yields a divergent integral. It is usually addressed by dimensional regularization, which breaks the superconformal symmetry. The authors found that a particular choice of the deformation parameters regularizes this integral. This observation initiated the hope that the deformations might serve as symmetry preserving regulators. However, while this deformed on-shell diagram is indeed superconformally invariant, it violates the Yangian symmetry. Essentially, the reason is that only the parameters c_i but not the $\hat{v}_i$ of the three-particle amplitudes were matched when gluing them together. Notice that the $\hat{v}_i$ do not enter in the amplitudes (1.50) and (1.52) but only in the Yangian generators. This is not satisfactory because to have a chance of exploiting the integrable structure for the computation of loop amplitudes, a regulator that preserves Yangian symmetry should be essential.

The study of deformed on-shell diagrams was continued in [108]. Undeformed on-shell diagrams can be labeled by permutations [100], as briefly mentioned at the end of the previous section. The authors of [108] found that the Yangian invariance of the deformed diagrams can be maintained by imposing a clever constraint on the deformation parameters that is formulated in terms of the associated permutation. With this technology they reexamined deformations of an on-shell diagram for the one-loop amplitude $\mathcal{A}^{(1)}_{4,2}$. They found that the Yangian invariant deformation vanishes for generic deformation parameters. There are only highly singular contributions for vanishing deformation parameters. Further studies of deformed on-shell diagrams for one-loop amplitudes were performed in [109]. Apart from this, [108] contains an investigation of deformations of the NMHV amplitude $\mathcal{A}^{(\text{tree})}_{6,3}$. These are of interest because $\mathcal{A}^{(1)}_{4,2}$ can be obtained from this amplitude as a so-called forward limit, where two legs are identified. Hence, they might provide an alternative approach to regulate the one-loop amplitude $\mathcal{A}^{(1)}_{4,2}$. We already know from the previous section that the three terms in the formula (1.26) for $\mathcal{A}^{(\text{tree})}_{6,3}$ can be understood as residues. Furthermore, each term is associated with an on-shell diagram which can be deformed individually. Demanding that these three deformed diagrams are compatible, i.e. that the parameters c_i and $\hat{v}_i$ for all diagrams agree, and that all superhelicities take the physical value $c_i = 0$ results in constraints which are only satisfied by the undeformed amplitude.

Does this result imply that there is no deformation of $\mathcal{A}^{(\text{tree})}_{6,3}$? In [110] it was argued that it does not suffice to deform the three on-shell diagrams contributing to the amplitude individually. Instead, one should take seriously that these diagrams are residues originating from a one-dimensional contour integral, which arises upon the evaluation of the Graßmannian integral (1.43). This line of thought led to a *deformed Graßmannian integral* [110, 111][8]

$$\mathcal{A}^{(\text{def.})}_{N,K} = \int \mathrm{d}\mathcal{C} \frac{\delta_*^{2(N-K)|0}(C^\perp \boldsymbol{\lambda})\delta_*^{2K|0}(C\tilde{\boldsymbol{\lambda}})\delta^{0|4K}(C\boldsymbol{\eta})}{(1,\ldots,K)^{1+\hat{v}_K^- - \hat{v}_1^+}\cdots(N,\ldots,K-1)^{1+\hat{v}_{K-1}^- - \hat{v}_N^+}} \,. \tag{1.54}$$

[8] In [110] the result is presented in supertwistor variables. The spinor helicity formula (1.54) can formally, which essentially means for the signature $(2,2)$, be obtained by applying Witten's half Fourier transform [67]. However, compared to equation (13) of [110] the variables $\hat{v}_i^+$ and $\hat{v}_i^-$ are exchanged in (1.54). The reason is a small typo in that $-c_i$ and not c_i is the eigenvalue of the superhelicity operator given in (11) of that reference. This can be verified by evaluating (16) therein for the MHV case.

Here the exponents are defined by [108]

$$\hat{v}_i^{\pm} = \hat{v}_i \pm \frac{c_i}{2} \,. \tag{1.55}$$

To ensure Yangian invariance they have to satisfy

$$\hat{v}_{i+K}^{-} = \hat{v}_i^{+} \tag{1.56}$$

for $i = 1, \dots, N$, where we count modulo N. Actually, (1.56) is the clever constraint on the deformation parameters found in [108], which we referred to in the previous paragraph. Writing the subscript on the left hand site as $\sigma(i) = i+K$, we see a permutation σ emerging. This particular permutation is a cyclic shift, which is associated with a special class of on-shell diagrams called *top-cells* in the language of [100].

The next step is to evaluate the deformed Graßmannian integral (1.54). The sample invariants $\mathcal{A}_{3,1}^{(\text{def.})}$, $\mathcal{A}_{3,2}^{(\text{def.})}$ and $\mathcal{A}_{4,2}^{(\text{def.})}$ discussed above can easily be obtained from (1.54) by solving for the bosonic delta functions. In this way, we also obtain the deformed MHV amplitudes [110, 111]

$$\mathcal{A}_{N,2}^{(\text{def.})} = \frac{\delta^{4|0}(P)\delta^{0|8}(Q)}{\langle 12\rangle^{1+\hat{v}_2^- - \hat{v}_1^+} \cdots \langle N1\rangle^{1+\hat{v}_1^- - \hat{v}_N^+}} \,. \tag{1.57}$$

Let us note a subtlety at this point. Because the deformations parameters $\hat{v}_i^{\pm}$ are complex, these deformed amplitudes are multi-valued functions in the spinors $\lambda^i, \tilde{\lambda}^i$. This multi-valuedness does not seem to cause problems in the MHV case. However, for deformed NMHV amplitudes, and in particular for $\mathcal{A}_{6,3}^{(\text{def.})}$, solving for the bosonic delta functions in (1.54) leaves us with a one-dimensional contour integral. Due to the complex exponents in (1.54), the residue theorem does not apply any longer for the evaluation of this integral. This may be viewed as an explanation why deforming the three individual residues in [108] did not succeed. Instead, one should find an appropriate closed contour taking into account the intricate structure of the multi-valued integrand. Partial results in this direction were obtained in [110]. Yet ultimately the problem of finding an appropriate contour that yields a closed form expression of the deformed six-particle NMHV amplitude $\mathcal{A}_{6,3}^{(\text{def.})}$ remained open in this reference. It is one of the issues addressed in this thesis.

We conclude this section with some remarks. First, in [110] close relations between the deformed Graßmannian integral (1.54) and the theory of multivariate hypergeometric functions [112] were observed, see also the books [113] and [114].

A further remark concerns an interesting construction of deformed amplitudes that was introduced in [115, 116]. Simple sample amplitudes were obtained by acting with a number of special solutions of a Yang-Baxter equation on a vacuum state. This method was explored systematically in [3, 117]. It led to a construction of all on-shell diagrams that are relevant for tree-level amplitudes. What is more, in [110] it is argued that it can be used to derive the deformed Graßmannian integral (1.54). We chose not to present this method here in detail mainly for two reasons. First, although the central object is a solution of a Yang-Baxter equation, its proper interpretation remains somewhat unclear. In particular, this solution does not commute with the central elements of the symmetry algebra $\mathfrak{gl}(4|4)$, see equation (102) in [3]. Thus it is not a usual $\mathfrak{gl}(4|4)$ invariant R-matrix. Second, this solution is typically represented as a formal integral operator. The proper contour of integration is unknown at present. We believe that finding the contour for this operator might be even more involved than finding directly the proper contour for the deformed Graßmannian integral formula (1.54). That is because to obtain this formula the integral operator has to be applied multiple times, probably each time with a different contour. Addressing these points would clearly be desirable.

As a final comment, an integrability-related approach to planar $\mathcal{N}=4$ SYM scattering amplitudes was also initiated in [118]. It is distinct from the Yangian invariant deformations outlined in this section and aims at computing amplitudes at finite coupling, see e.g. the recent work [119] for a comprehensive list of references.

1.4 Objectives and Outline

The long-term objective of the line of research presented in this thesis is to make the powerful methods of integrable models available for the computation of scattering amplitudes in planar $\mathcal{N}=4$ SYM, even at all-loop level. Thereby it may be possible to repeat the tremendous success story of integrability in the spectral problem. Moreover, we believe that such a development would help to clarify the integrable structure that seems to underlie the planar $\mathcal{N}=4$ model as a whole.

In this thesis we undertake steps in this direction by building a bridge between tree-level scattering amplitudes on one side and integrable models, as they appear e.g. in the one-loop spectral problem, on the other side. The reader might argue that the relation of these amplitudes to integrability was already clarified by showing their Yangian invariance as discussed in the preceding section. However, this pioneering connection alone did not immediately enable the application of the techniques associated with integrable systems to the calculation of amplitudes. In the present thesis we supplement this work by an effort for a robust construction whose algebraic and representation theoretic foundations we lay in chapter 2. Based on this groundwork we start erecting our bridge on both riverbanks. In chapter 3 we work on the integrable systems side employing the mindset of sections 1.1 and 1.2. We show that a Bethe ansatz can be applied for the computation of certain Yangian invariants. Our work continues in chapter 4 on the amplitudes side, where we make use of the ideas presented in section 1.3. Here we develop a second method for the computation of Yangian invariants which is based on the Graßmannian integral approach. These two methods are in a sense complementary but they also have an overlap, see figure 1.2 and the explanations in the following paragraphs. Interestingly, this might make it possible to use our bridge also in the reverse direction and thus apply ideas from amplitudes to gain new insights into integrable models as such. Finally, in chapter 5 we assess the status of our constructions and plan future steps.

Let us outline our work on a more technical level. The key principle on which our investigations are based on is the Yangian invariance of tree-level amplitudes. In **chapter 2** we start by reviewing the algebraic foundation of the Yangian of the Lie superalgebra $\mathfrak{gl}(n|m)$ in the QISM language, which we encountered in section 1.1 in connection with integrable spin chains. What is more, we formulate the *Yangian invariance condition within the QISM*. This will later allow us to use the QISM framework for the construction of Yangian invariants. We move on by discussing a class of unitary representations of the non-compact superalgebra $\mathfrak{u}(p,q|m)\subset\mathfrak{gl}(n=p+q|m)$ that are built in terms of harmonic oscillator algebras. For $\mathfrak{u}(2,2|4)$ essentially these representations are frequently used in the $\mathcal{N}=4$ SYM spectral problem. Moreover, they are equivalent to the spinor helicity variables employed for amplitudes in section 1.3. The compact $\mathfrak{u}(n)$ case is of interest to make contact with the common literature on spin chains. Thus this class of representations provides a great versatility. Working with $\mathfrak{u}(p,q|m)$ instead of directly focusing on the amplitude case $\mathfrak{u}(2,2|4)$ also satisfies our mathematical curiosity to systematically explore the notion of Yangian invariance. We conclude by discussing some sample Yangian invariants to gain

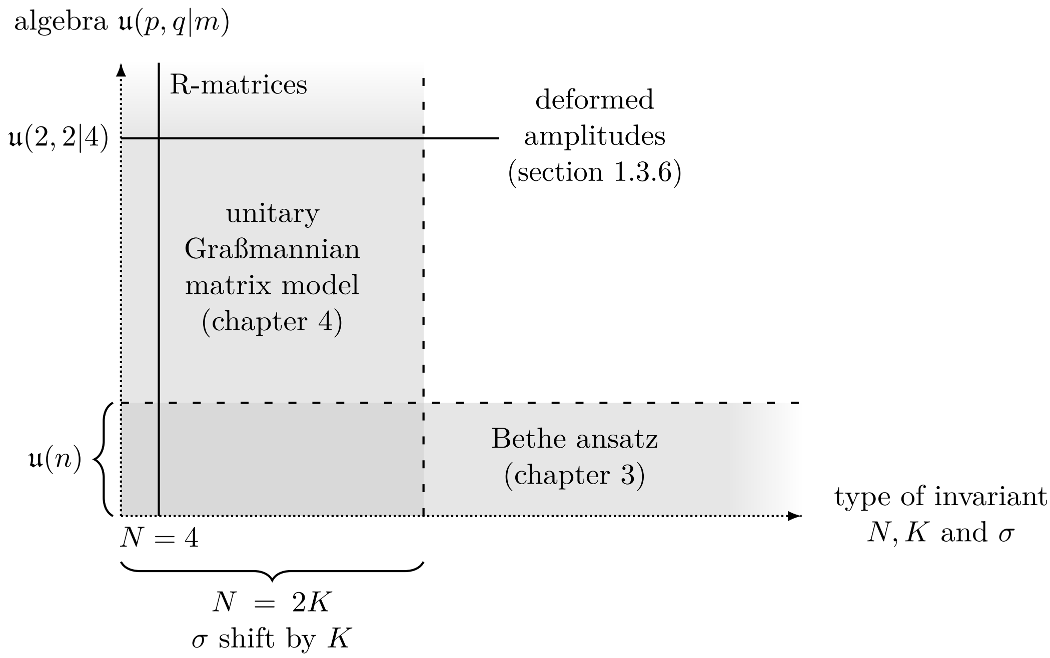


Figure 1.2: Schematic view of the "landscape" of Yangian invariants. The Bethe ansatz is applicable for the construction of Yangian invariants for compact algebras $\mathfrak{u}(n)$, the details being worked out for $\mathfrak{u}(2)$. In this setting it leads to a classification of all invariants. The unitary Graßmannian matrix model can be used for the large class of non-compact superalgebras $\mathfrak{u}(p,q|m)$. However, at the moment it is restricted to a certain type of invariants. In this spirit both methods are complementary.

more familiarity with the representations. In doing so we notice the necessity of using a further family of oscillator representations, which are dual to the ordinary ones. The Yangian invariants that we study are associated with a spin chain of N sites out of which K carry a dual representation. We observe that for $(N,K) = (4,2)$ the Yangian invariance condition is equivalent to the Yang-Baxter equation, whose solutions are called R-matrices. The foundations laid in this chapter are used throughout the thesis.

In **chapter 3** we make use of the QISM for the construction of Yangian invariants. To this end, we first review the Bethe ansatz for inhomogeneous $\mathfrak{u}(2)$ spin chains in the QISM setting. This class of spin chains includes the Heisenberg model, which served as an example in section 1.1. These preparations allow us to develop a *Bethe ansatz for Yangian invariants* with $\mathfrak{u}(2)$ representations by identifying these invariants with specific eigenstates of particular spin chains. These eigenstates are characterized by a special case of the usual Bethe equations. Unlike the full equations, this special case can easily be solved explicitly. Doing so we recover some of the sample invariants from chapter 2. Furthermore, all solutions of the equations can be classified in terms of permutations σ. These are analogous to the permutations appearing for deformed amplitudes in section 1.3.6. Hence the compact $\mathfrak{u}(2)$ Yangian invariants constructed in this chapter can be considered as toy models for amplitudes. Their representation labels and inhomogeneities can be identified with the deformation parameters of those amplitudes. Some additional material on the Bethe ansatz for Yangian invariants, in particular concerning the extension to $\mathfrak{u}(n)$, is deferred to **appendix A**. On a different note, we show that Yangian invariants for $\mathfrak{u}(n)$ can be interpreted as partition functions of certain vertex models on in general non-rectangular lattices. This allows us to view our Bethe ansatz construction of Yangian invariants as a vast generalization of Baxter's little known perimeter Bethe ansatz.

Chapter 4 is devoted to an alternative construction of Yangian invariants. Taking inspiration from the Graßmannian integral for deformed amplitudes of section 1.3.6, we derive a Graßmannian integral formula for a family of Yangian invariants with $N = 2K$ and oscillator representations of the non-compact superalgebra $\mathfrak{u}(p, q|m)$. A key difference to the original formula is that for these representations the formal delta function in the integrand gets replaced by an exponential function of oscillators. Furthermore, we are able to fix the contour of integration to be a unitary group manifold. For special values of the deformation parameters this integral reduces to a well-known unitary matrix model due to Brezin, Gross and Witten. Consequently, we term our formula *unitary Graßmannian matrix model*. Evaluating it for special cases, we once again rederive sample invariants from chapter 2. To relate the invariants obtained from this integral to deformed amplitudes, we apply a change of basis mapping the oscillator variables to spinor helicity-like variables of the algebra $\mathfrak{u}(p, p|m)$. In particular this provides us with a proposal for a refined Graßmannian integral formula for deformed amplitudes when specializing to $\mathfrak{u}(2, 2|4)$. This formula features several improvements over the one in section 1.3.6. Importantly, it is formulated in the physical Minkowski signature. In addition, we find that the unitary contour circumvents all issues with branch cuts pointed out in the aforementioned section. Finally, we put our refined formula to the test. We are able to rederive the already known deformed amplitude $\mathcal{A}_{4,2}^{(\text{def.})}$. What is more, we obtain a natural candidate for the presently unknown $\mathcal{A}_{6,3}^{(\text{def.})}$. Curiously, there is a caveat because in the undeformed limit the tree-level amplitude $\mathcal{A}_{6,3}^{(\text{tree})}$ emerges only in a certain region of the whole momentum space. Let us add that supplementary material on the unitary Graßmannian integral is presented in **appendix B**.

The final **chapter 5** contains our conclusions. Furthermore, we provide an outlook on possibilities to extend our work. In particular, we ponder about the implications of the just mentioned caveat on the symmetries of tree-level amplitudes.

Chapter 2

Yangians and Representations

The purpose of this chapter is to set up a common framework within which we will develop methods for the construction of Yangian invariants in the following chapters 3 and 4.

We begin by reviewing the Yangian of the Lie superalgebra $\mathfrak{gl}(n|m)$ in section 2.1. In contrast to the brief discussion of the Yangian algebra in the introductory section 1.3, we choose a QISM formulation that highlights the connections with integrable systems, in particular with the Yang-Baxter equation and spin chains. In section 2.2 we translate the important Yangian invariance condition into this language. Before we are able to explore solutions of this equation, we have to select representations of the $\mathfrak{gl}(n|m)$ algebra. Therefore, in section 2.3 we present a class of unitary representations of $\mathfrak{u}(p,q|m) \subset \mathfrak{gl}(n|m)$, that are constructed from bosonic and fermionic oscillators. For $\mathfrak{u}(2,2|4)$ these turn out to be equivalent to the spinor helicity variables of section 1.3. The oscillator formalism allows us to work with more general non-compact superalgebras.

We continue in section 2.4 by "manually" constructing sample solutions of the Yangian invariance condition that are associated with spin chains consisting of two, three and four sites. Initially we restrict to sample invariants with representations of the compact bosonic algebra $\mathfrak{u}(n)$. These will be recovered systematically by means of a Bethe ansatz in chapter 3. Then we attempt to generalize the sample invariants to the full non-compact supersymmetric $\mathfrak{u}(p,q|m)$ setting. A Graßmannian integral construction for such Yangian invariants will be developed in chapter 4. Furthermore, we explain in section 2.4 that for the four-site sample invariant the Yangian invariance condition is equivalent to a Yang-Baxter equation. Hence for the algebra $\mathfrak{u}(2,2|4)$ this invariant is essentially the R-matrix of the integrable spin chain governing the planar $\mathcal{N}=4$ SYM one-loop spectral problem. Later in chapter 4 we will apply a change of basis from oscillators to spinor helicity variables that turns this invariant into the amplitude $\mathcal{A}_{4,2}^{(\text{tree})}$.

2.1 Yangian Algebra

We encountered the Yangian of $\mathfrak{sl}(4|4)$ as an infinite-dimensional extension of the complexified superconformal algebra in the review of $\mathcal{N}=4$ SYM amplitudes in section 1.3.4. Here we provide a systematic account on the *Yangian of the Lie superalgebra* $\mathfrak{gl}(n|m)$. Yangian algebras that are based on bosonic Lie algebras, such as $\mathfrak{gl}(n)$, were introduced by Drinfeld [84]. In physics they occur in the study of integrable models, cf. [19, 18]. Moreover, they are of interest mathematically as prominent examples of a special class of Hopf algebras called "quantum groups" [120, 121], see also [122]. There exist different formulations of Yangian algebras, one of which we saw in section 1.3.4. Here we pursue a more elaborate approach that has its origins in the study of integrable models, in particular spin chains.

This is the *quantum inverse scattering method* (QISM) mentioned already in section 1.1. It goes back to work of the Leningrad school around Faddeev in the 1970s and 80s. It explores the algebraic and representation theoretic consequences of the Yang-Baxter equation and provides a toolbox to "solve" integrable models. Authoritative reviews of the QISM can be found in [14, 15]. See also [123] for an alternative selection of topics and [17] for a historic perspective. The formulation of Yangians within the QISM and its representation theory are discussed extensively in [20]. The presentation in this section is to a large extent influenced by this reference. The works cited so far concentrate on bosonic algebras. A $\mathfrak{gl}(n|m)$ version of the QISM can be found in [124]. The Yangian of this superalgebra was defined in [125], see also [126]. In this section we immediately present the formulas for the $\mathfrak{gl}(n|m)$ case. The $\mathfrak{gl}(n)$ analogues can be easily obtained by dropping the grading factors $(-1)^{|\cdots|}$.

To begin with, we recapitulate some basic notions of the super vector space $\mathbb{C}^{n|m}$ and introduce the general linear Lie superalgebra $\mathfrak{gl}(\mathbb{C}^{n|m}) \equiv \mathfrak{gl}(n|m)$. Most of these basics are nicely summarized in [124], see the references therein for more details. An extensive discussion of Lie superalgebras can be found e.g. in [127]. Let the vectors $e_{\mathcal{A}}$ with the superindex $\mathcal{A} = 1, \ldots, n+m$ form a basis of the super vector space $\mathbb{C}^{n|m}$. We allow for a non-standard grading by choosing the integer partitions $n = p + q$ and $m = r + s$. Based thereon we assign the degree

$$|e_{\mathcal{A}}| = |\mathcal{A}| = \begin{cases} 0 & \text{for} \quad \mathcal{A} = \phantom{p+r+q+{}} 1, \ldots, p\,, \\ 1 & \text{for} \quad \mathcal{A} = \phantom{p+r+{}} p+1, \ldots, p+r\,, \\ 0 & \text{for} \quad \mathcal{A} = \phantom{p+{}} p+r+1, \ldots, p+r+q\,, \\ 1 & \text{for} \quad \mathcal{A} = p+r+q+1, \ldots, p+r+q+s\,. \end{cases} \tag{2.1}$$

This freedom in the choice of the grading is already adapted to the representations of $\mathfrak{u}(p,q|r+s) \subset \mathfrak{gl}(n|m)$, which we will study in section 2.3 below. We define supermatrices $E_{\mathcal{AB}}$ by $E_{\mathcal{AB}}\, e_{\mathcal{C}} = \delta_{\mathcal{BC}}\, e_{\mathcal{A}}$. They satisfy

$$E_{\mathcal{AB}} E_{\mathcal{CD}} = \delta_{\mathcal{BC}} E_{\mathcal{AD}} \tag{2.2}$$

and are of degree $|E_{\mathcal{AB}}| = |\mathcal{A}| + |\mathcal{B}|$. The $E_{\mathcal{AB}}$ can be considered as generators of the defining representation of the *general linear Lie superalgebra* $\mathfrak{gl}(n|m)$. In this context the super vector space these generators act on is denoted by $\square = \mathbb{C}^{n|m}$. The $\mathfrak{gl}(n|m)$ algebra is defined by the relations

$$[J_{\mathcal{AB}}, J_{\mathcal{CD}}\} = \delta_{\mathcal{CB}} J_{\mathcal{AD}} - (-1)^{(|\mathcal{A}|+|\mathcal{B}|)(|\mathcal{C}|+|\mathcal{D}|)} \delta_{\mathcal{AD}} J_{\mathcal{CB}} \tag{2.3}$$

for the generators $J_{\mathcal{AB}}$ of degree $|\mathcal{A}| + |\mathcal{B}|$. Here we employed the graded commutator

$$[U, V\} = UV - (-1)^{|U||V|} VU \tag{2.4}$$

for homogeneous elements $U, V \in \mathfrak{gl}(n|m)$. We consider a representation of this algebra where the generators $J_{\mathcal{AB}}$ act as operators on some super vector space $\mathcal{V}$. In slight abuse of notation we often refer to $\mathcal{V}$ itself as representation. Operators acting on the tensor product $\mathcal{V} \otimes \mathcal{V}'$ of two super vectors spaces obey

$$(U \otimes V)(U' \otimes V') = (-1)^{|V||U'|} UU' \otimes VV'\,. \tag{2.5}$$

At this point, we discus some automorphisms of the superalgebra $\mathfrak{gl}(n|m)$ given in (2.3), which we will make use of later. One readily verifies that

$$J_{\mathcal{AB}} \mapsto -(-1)^{|\mathcal{A}|+|\mathcal{A}||\mathcal{B}|} J_{\mathcal{AB}}^{\dagger} \tag{2.6}$$

is an automorphism. Here the conjugation satisfies $(UV)^\dagger = V^\dagger U^\dagger$ and $(U^\dagger)^\dagger = U$.[1] Notice that for bosonic algebras the map (2.6) is an antiinvolution. A similar but distinct automorphism of the $\mathfrak{gl}(n|m)$ algebra (2.3) is

$$J_{\mathcal{AB}} \mapsto -(-1)^{|\mathcal{A}|+|\mathcal{A}||\mathcal{B}|} J_{\mathcal{BA}} \,. \tag{2.7}$$

We will encounter (2.6) and (2.7) in section 2.3 in the context of so-called dual representations. Yet another automorphism of (2.3) is

$$J_{\mathcal{AB}} \mapsto J_{\mathcal{AB}} + v\delta_{\mathcal{AB}}(-1)^{|\mathcal{B}|} \tag{2.8}$$

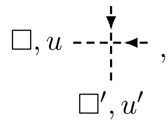
with an arbitrary parameter $v \in \mathbb{C}$.

After introducing $\mathfrak{gl}(n|m)$, we move on to discuss the *Yangian* [84] of this Lie superalgebra. It is defined by the relation

$$R_{\square\square'}(u-u')(M(u)\otimes 1)(1\otimes M(u')) = (1\otimes M(u'))(M(u)\otimes 1)R_{\square\square'}(u-u') \,. \tag{2.9}$$

In what follows we explain this definition. The *R-matrix*

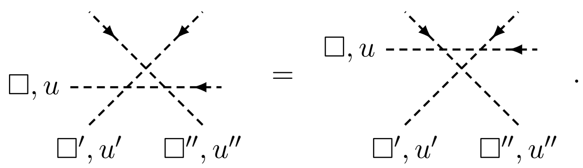
$$R_{\square\square'}(u-u') = 1 + (u-u')^{-1}\sum_{\mathcal{A},\mathcal{B}} E_{\mathcal{AB}} \otimes E'_{\mathcal{BA}}(-1)^{|\mathcal{B}|} = \tag{2.10}$$



acts on two copies of the defining representation $\square \otimes \square' = \mathbb{C}^{n|m} \otimes \mathbb{C}^{n|m}$, see e.g. [124]. In the bosonic case it is attributed to Yang. The prime at the defining generator $E'_{\mathcal{BA}}$ merely emphasizes that it acts on the space $\square'$. The two spaces also enter the definition of the Yangian in (2.9). They are called *auxiliary spaces* and in the graphical notation established in (2.10) we associate a dashed line with each of them. The R-matrix depends on the complex *spectral parameters* u and u', each one belonging to one of the representation spaces. They are also included in the graphical representation. Note that the identity operator on $\mathbb{C}^{n|m}$ may be written as $\sum_{\mathcal{A}} E_{\mathcal{AA}}$. The 1 in (2.10) stands for the appropriate identity operator on the tensor product. The R-matrix (2.10) is a solution of the *Yang-Baxter equation*

$$\begin{aligned} &R_{\square\square'}(u-u')R_{\square\square''}(u-u'')R_{\square'\square''}(u'-u'') \\ &= R_{\square'\square''}(u'-u'')R_{\square\square''}(u-u'')R_{\square\square'}(u-u') \,, \end{aligned} \tag{2.11}$$

which acts in the tensor product $\square \otimes \square' \otimes \square''$. Graphically it reads



(2.12)

The arrows of the lines define an orientation which translates into the order in which the R-matrices act. R-matrices "earlier" on the line are right of "later" ones in the corresponding formula. Comparing with (2.11), we may interpret the definition of the Yangian in (2.9) as a Yang-Baxter equation where the third space is left unspecified. The R-matrices that would act on this space are replaced by the operator valued *monodromy matrix* $M(u)$. This

[1] The superalgebra $\mathfrak{gl}(n|m)$ may also be equipped with a graded conjugation $\ddagger$ obeying $(UV)^\ddagger = (-1)^{|U||V|}V^\ddagger U^\ddagger$ and $(U^\ddagger)^\ddagger = (-1)^{|U|}U$, cf. [128]. Then $J_{\mathcal{AB}} \mapsto -J^\ddagger_{\mathcal{AB}}$ defines an automorphism.

matrix contains the infinitely many generators $M^{(l)}_{\mathcal{AB}}$ with $l = 1, 2, 3, \ldots$ of the Yangian. They are obtained from an expansion in the spectral parameter

$$M(u) = \sum_{\mathcal{A},\mathcal{B}} E_{\mathcal{AB}} M_{\mathcal{AB}}(u)(-1)^{|\mathcal{B}|}\,, \quad M_{\mathcal{AB}}(u) = M^{(0)}_{\mathcal{AB}} + u^{-1} M^{(1)}_{\mathcal{AB}} + u^{-2} M^{(2)}_{\mathcal{AB}} + \ldots \tag{2.13}$$

with the normalization

$$M^{(0)}_{\mathcal{AB}} = \delta_{\mathcal{AB}}(-1)^{|\mathcal{B}|}. \tag{2.14}$$

The degree of $M_{\mathcal{AB}}(u)$ is $|\mathcal{A}| + |\mathcal{B}|$. With (2.13) the defining relation (2.9) is equivalent to

$$\begin{aligned}(u' - u)&[M_{\mathcal{AB}}(u), M_{\mathcal{CD}}(u')\}\\ &= (M_{\mathcal{CB}}(u) M_{\mathcal{AD}}(u') - M_{\mathcal{CB}}(u') M_{\mathcal{AD}}(u))(-1)^{|\mathcal{A}||\mathcal{D}|+|\mathcal{A}||\mathcal{C}|+|\mathcal{C}||\mathcal{D}|}\,.\end{aligned} \tag{2.15}$$

After expanding the monodromy elements this reads

$$\begin{aligned}&[M^{(k)}_{\mathcal{AB}}, M^{(l)}_{\mathcal{CD}}\}\\ &\quad= \sum_{q=1}^{\min(k,l)} (M^{(k+l-q)}_{\mathcal{CB}} M^{(q-1)}_{\mathcal{AD}} - M^{(q-1)}_{\mathcal{CB}} M^{(k+l-q)}_{\mathcal{AD}})(-1)^{|\mathcal{A}||\mathcal{D}|+|\mathcal{A}||\mathcal{C}|+|\mathcal{D}||\mathcal{C}|}\,.\end{aligned} \tag{2.16}$$

These quadratic relations provide an explicit definition of the Yangian algebra (2.9) in terms of its generators $M^{(l)}_{\mathcal{AB}}$.

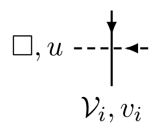
We remark that (2.16) manifestly displays a filtration of the Yangian algebra, with regard to which the generators $M^{(l)}_{\mathcal{AB}}$ are of level l, cf. [20].[2] In addition, one easily deduces from (2.16) that all generators $M^{(l)}_{\mathcal{AB}}$ with $l > 2$ can be expressed via $M^{(1)}_{\mathcal{AB}}$ and $M^{(2)}_{\mathcal{AB}}$. By setting $r = s = 1$ we see that the $M^{(1)}_{\mathcal{BA}}$ satisfy the $\mathfrak{gl}(n|m)$ algebra (2.3). Furthermore, expanding (2.15) only in the spectral parameter u leads to

$$[M^{(1)}_{\mathcal{AB}}, M_{\mathcal{CD}}(u')\} = \delta_{\mathcal{AD}} M_{\mathcal{CB}}(u') - (-1)^{(|\mathcal{A}|+|\mathcal{C}|)(|\mathcal{B}|+|\mathcal{D}|)} \delta_{\mathcal{CB}} M_{\mathcal{AD}}(u')\,. \tag{2.17}$$

Thus the monodromy elements transform in the adjoint representation of $\mathfrak{gl}(n|m)$.

Having introduced the Yangian, we study realizations of its defining relation (2.9) where the generators $M_{\mathcal{AB}}(u)$ act on the tensor product $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_N$ of $\mathfrak{gl}(n|m)$ representations. This space is referred to as *quantum space*. Let us introduce the *Lax operator*

$$R_{\square\mathcal{V}_i}(u - v_i) = f_{\mathcal{V}_i}(u - v_i)\left(1 + (u - v_i)^{-1} \sum_{\mathcal{A},\mathcal{B}} E_{\mathcal{AB}} J^i_{\mathcal{BA}}(-1)^{|\mathcal{B}|}\right) = \tag{2.18}$$

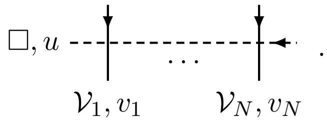


acting on the tensor product $\square \otimes \mathcal{V}_i$ of an auxiliary and a local quantum space, which we indicate graphically by a solid line. This Lax operator is characterized as the solution of a Yang-Baxter equation like (2.12), where the third space is replaced by $\mathcal{V}_i$. Hence it is already a solution of (2.9). Clearly this equation does not determine the scalar normalization $f_{\mathcal{V}_i}$. Note that up to a change in this normalization the complex inhomogeneity parameter v_i can be altered by applying the $\mathfrak{gl}(n|m)$ automorphism (2.8) to the generators $J^i_{\mathcal{BA}}$. Further

[2] This notion of "level" differs from the one mentioned in the context of scattering amplitudes right before (1.36).

solutions of (2.9) are obtained by multiplying multiple Lax operators acting in different quantum spaces,

$$M(u) = R_{\square\mathcal{V}_1}(u - v_1) \cdots R_{\square\mathcal{V}_N}(u - v_N) = \tag{2.19}$$



This can be proved using the so-called "train argument" which makes use of the Yang-Baxter equations for the individual Lax operators. Notice that (2.14) imposes a condition on the normalizations $f_{\mathcal{V}_i}$ in the Lax operators. It is readily satisfied by setting $f_{\mathcal{V}_i} = 1$. However, at times we will use non-trivial normalization factors. The monodromy matrix in (2.19) is that of an *inhomogeneous spin chain* with N sites. Here the meaning of the word "inhomogeneous" is twofold. First, we associate a inhomogeneity v_i with each site. Second, each site carries a different representation $\mathcal{V}_i$. The quantum space is the state space of the spin chain. In section 3.1 we will discuss how the Heisenberg spin chain can be obtained from such a monodromy matrix. From now on we focus on realizations of the Yangian with a monodromy of the form (2.19).

Expanding the monodromy (2.19) according to (2.13) leads to expressions for the Yangian generators in terms of $\mathfrak{gl}(n|m)$ generators. With $f_{\mathcal{V}_i} = 1$ this results in

$$M^{(1)}_{\mathcal{AB}} = \sum_{i=1}^{N} J^i_{\mathcal{BA}}\,, \quad M^{(2)}_{\mathcal{AB}} = \sum_{i=1}^{N} v_i J^i_{\mathcal{BA}} + \sum_{\substack{i,j=1\\ i<j}}^{N} \sum_{\mathcal{C}} (-1)^{|\mathcal{C}|} J^j_{\mathcal{BC}} J^i_{\mathcal{CA}}\,, \quad \ldots\,. \tag{2.20}$$

Let us also discuss a different way of expanding the monodromy elements,

$$\begin{aligned} M(u) &= 1 + u^{-1} M^{(1)} + u^{-2} M^{(2)} + \ldots \\ &= \exp\left(u^{-1} M^{[1]} + u^{-2} M^{[2]} + \ldots\right). \end{aligned} \tag{2.21}$$

The matrix elements $M^{[l]}_{\mathcal{AB}}$ of the new expansion coefficients $M^{[l]}$ are defined analogously to those of the original coefficients $M^{(l)}$ in (2.13). We obtain

$$M^{[1]} = M^{(1)}\,, \quad M^{[2]} = M^{(2)} - \frac{1}{2} M^{(1)} M^{(1)}\,, \quad \ldots\,. \tag{2.22}$$

Using (2.20) this allows us to compute the explicit form of the generators also for this expansion,

$$\begin{aligned} M^{[1]}_{\mathcal{AB}} &= \sum_{i=1}^{N} J^i_{\mathcal{BA}}\,, \\ M^{[2]}_{\mathcal{AB}} &= \sum_{i=1}^{N} \left(v_i J^i_{\mathcal{BA}} - \frac{1}{2} \sum_{\mathcal{C}} (-1)^{|\mathcal{A}||\mathcal{B}| + |\mathcal{A}||\mathcal{C}| + |\mathcal{B}||\mathcal{C}|} J^i_{\mathcal{CA}} J^i_{\mathcal{BC}} \right) \\ &\quad + \frac{1}{2} \sum_{\substack{i,j=1\\ i<j}}^{N} \sum_{\mathcal{C}} (-1)^{|\mathcal{C}|} \left(J^j_{\mathcal{BC}} J^i_{\mathcal{CA}} - J^i_{\mathcal{BC}} J^j_{\mathcal{CA}} \right), \\ &\ldots\,. \end{aligned} \tag{2.23}$$

This is precisely the form of the Yangian generators which we encountered in (1.34) and (1.36) in the discussion of $\mathcal{N} = 4$ SYM scattering amplitudes.

The definition (2.9) of the Yangian in terms of a monodromy matrix $M(u)$ is often called RTT-relation.[3] The generators $M^{[l]}_{\mathcal{AB}}$ that we obtained from the expansion in (2.21) are essentially what is known as Drinfeld's first realization of the Yangian [84]. We did not discuss Drinfeld's second realization [129]. It is based on a Gauß decomposition of the monodromy matrix $M(u)$ into the product of an upper triangular, a diagonal and a lower triangular matrix. The generators of the second realization are obtained from an expansion of the elements of these matrices in the spectral parameter, c.f. [130]. Let us add that in this section we did not emphasize the quantum group and in particular the Hopf algebra structure of the Yangian. It can be nicely phrased in the QISM language employed here, see e.g. [122].

2.2 Yangian Invariance

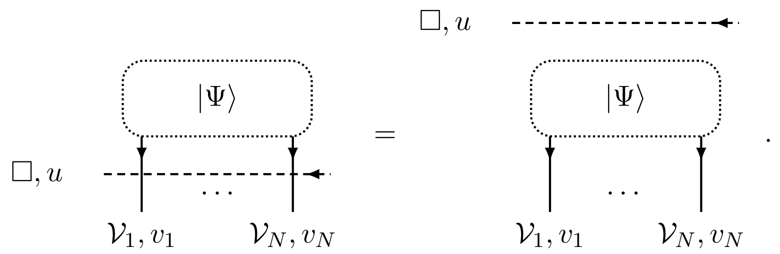
Recall from sections 1.3.4 and 1.3.6 that tree-level amplitudes $\mathcal{A}^{(\text{tree})}_{N,K}$ of $\mathcal{N}=4$ SYM and deformations $\mathcal{A}^{(\text{def.})}_{N,K}$ thereof are Yangian invariant. How does this translate into the QISM language? A state $|\Psi\rangle$ in the quantum space $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_N$ is *Yangian invariant* iff

$$M(u)|\Psi\rangle = |\Psi\rangle\,. \tag{2.24}$$

On the level of matrix elements this translates into[4]

$$M_{\mathcal{AB}}(u)|\Psi\rangle = (-1)^{|\mathcal{B}|}\delta_{\mathcal{AB}}|\Psi\rangle\,, \tag{2.25}$$

where we used (2.13). With the help of (2.19) we represent (2.24) graphically as



(2.26)

Here the dashed line on the right hand side corresponds to an identity operators acting on the auxiliary space $\square$. The invariant $|\Psi\rangle$ itself is symbolized by a dotted "black box" without specifying the interior. During the course of this thesis we will construct solutions of the Yangian invariance condition and so to speak learn about the structure of this interior. In order to make contact with the notion of Yangian invariance used in sections 1.3.4 and 1.3.6, we expand (2.24) in the spectral parameter. Using the expansion in the first line of (2.21) we obtain

$$M^{(l)}_{\mathcal{AB}}|\Psi\rangle = 0 \tag{2.27}$$

for all $l = 1, 2, 3, \ldots$. The expansion in the second line of (2.21) yields

$$M^{[l]}_{\mathcal{AB}}|\Psi\rangle = 0 \tag{2.28}$$

[3]The name has its origin in the frequent use of the symbol "$T(u)$" for the monodromy $M(u)$ in the literature.

[4]Analogous equations were shown to be satisfied by the physical vacuum state of integrable two-dimensional quantum field theories in [131, 132].

for all $l = 1, 2, 3, \ldots$. Each set of conditions, (2.27) as well as (2.28), provides a definition of Yangian invariance that is equivalent to (2.24). They show that $|\Psi\rangle$ is a one-dimensional representation of the Yangian as it is annihilated by all its generators. A further simplification is possible. Arguing as in the paragraph after (2.16), one shows that if (2.27) holds for $l = 1$ and $l = 2$, it is satisfied for all l. The same is true for (2.28). This is essentially the definition of Yangian invariance used for the amplitudes $\mathcal{A}^{(\text{tree})}_{N,K}$ in (1.35) and (1.40), and for the deformed amplitudes $\mathcal{A}^{(\text{def.})}_{N,K}$ in (1.47). Thus we know already something about the "black box" $|\Psi\rangle$ in (2.26). For the representation (1.30) of $\mathfrak{sl}(4|4)$ in terms of spinor helicity variables, it may be identified with the amplitudes $\mathcal{A}^{(\text{tree})}_{N,K}$ or the deformed amplitudes $\mathcal{A}^{(\text{def.})}_{N,K}$. While the definition of Yangian invariance in (2.28) allows to make contact with the literature on scattering amplitudes, let us emphasize the advantage of (2.24). It represents the Yangian invariance condition as an eigenvalue problem for monodromy matrix elements and thus makes the powerful QISM toolbox applicable. In particular, the formulation (2.24) will be exploited in section 3.2, where this equation is solved using an algebraic Bethe ansatz.

We conclude with a comment on "Yangian invariance" or "Yangian symmetry" of spin chain models. In this context both terms are used synonymously and signify that the Hamiltonian of a model commutes with the generators of a Yangian algebra. The Hamiltonians of the original Heisenberg model with spin $\frac{1}{2}$ $\mathfrak{su}(2)$ symmetry and $\mathfrak{su}(n)$ generalizations thereof contain only nearest neighbor interactions. They can be derived from a monodromy of the form (2.19), cf. section 3.1 below. Nonetheless, the Yangian invariance is spoiled for a finite number of sites N by boundary terms, cf. [18]. For an infinite number of sites a Yangian symmetry was also discovered in the Hubbard model [133], see also [134]. Furthermore, for an infinite number of sites there are classes of integrable spin chains with long-range interactions that exhibit Yangian symmetry, see e.g. [135]. The Hubbard model as well as such long-range spin chains are of relevance for describing the multi-loop dilatation operator in planar $\mathcal{N} = 4$ SYM, see the review [45] and recall section 1.2. The Haldane-Shastry chain [136, 137] is a long-range model with Yangian symmetry even for a finite number of sites. A class of further models with Yangian invariance at finite length was investigated recently in [138], see also the references therein.

2.3 Oscillator Representations

In the previous sections we discussed the Yangian of $\mathfrak{gl}(n|m)$ and the Yangian invariance condition on an algebraic level. Here we introduce those representations of the $\mathfrak{gl}(n|m)$ algebra that we will employ at the sites of the spin chain monodromy (2.19) defining the Yangian. We work with certain classes of unitary representations of the non-compact algebra $\mathfrak{u}(p, q|m) \subset \mathfrak{gl}(p + q|m)$ that are constructed in terms of bosonic and fermionic oscillator algebras. Such representations have a long history in the physics literature and are sometimes referred to as "ladder representations", see e.g. [139] for the bosonic case. We follow the presentation of [140] which includes the generalization to superalgebras. Our primary interest for the study of Yangian invariants comes from planar $\mathcal{N} = 4$ SYM scattering amplitudes. As already mentioned above, for the algebra $\mathfrak{u}(2, 2|4)$ these oscillator representations are unitarily equivalent to the realization in terms of spinor helicity variables in (1.30). This equivalence will be shown below in section 4.2. For now, however, we discuss the oscillator representations in the general setting of the algebra $\mathfrak{u}(p, q|m)$.

The basic ingredient is a family of *superoscillators* obeying

$$[\mathbf{A}_{\mathcal{A}}, \bar{\mathbf{A}}_{\mathcal{B}}\} = \delta_{\mathcal{A}\mathcal{B}}\,, \quad [\mathbf{A}_{\mathcal{A}}, \mathbf{A}_{\mathcal{B}}\} = 0\,, \quad [\bar{\mathbf{A}}_{\mathcal{A}}, \bar{\mathbf{A}}_{\mathcal{B}}\} = 0\,, \tag{2.29}$$

where the indices of the annihilation operators $\mathbf{A}_{\mathcal{A}}$ and creation operators $\bar{\mathbf{A}}_{\mathcal{A}}$ take the values $\mathcal{A} = 1, \ldots, n+m$. It is equipped with a conjugation $\dagger$ and acts on a Fock space $\mathcal{F}$ that is spanned by monomials in $\bar{\mathbf{A}}_{\mathcal{A}}$ acting on a vacuum state $|0\rangle$,

$$\mathbf{A}_{\mathcal{A}}^{\dagger} = \bar{\mathbf{A}}_{\mathcal{A}}\,, \quad \mathbf{A}_{\mathcal{A}}|0\rangle = 0\,. \tag{2.30}$$

The $\mathfrak{gl}(n|m)$ algebra (2.3) can be realized as

$$\mathbf{J}_{\mathcal{A}\mathcal{B}} = \bar{\mathbf{A}}_{\mathcal{A}}\mathbf{A}_{\mathcal{B}}\,. \tag{2.31}$$

From now on, we mark $\mathfrak{gl}(n|m)$ generators which are realized in terms of oscillators by bold letters. This construction yields unitary representations of the *compact* algebra $\mathfrak{u}(n|m) \subset \mathfrak{gl}(n|m)$, cf. [140].

We proceed to oscillator representations of the *non-compact* algebra $\mathfrak{u}(p,q|m)$. For this we split the family of superoscillators with $\mathfrak{gl}(n|m)$ index $\mathcal{A}$ into two parts. One carries a $\mathfrak{gl}(p|r)$ index $\mathsf{A} = 1, \ldots, p{+}r$ and the other one a $\mathfrak{gl}(q|s)$ index $\dot{\mathsf{A}} = p{+}r{+}1, \ldots, p{+}r{+}q{+}s$ with $p+q=n$ and $r+s=m$. The degrees $|\mathsf{A}|$ and $|\dot{\mathsf{A}}|$ of these indices can be inferred from $|\mathcal{A}|$ specified in (2.1). For the annihilation operators this reads

$$(\mathbf{A}_{\mathcal{A}}) = \begin{pmatrix} \mathbf{A}_{\mathsf{A}} \\ \hdashline \mathbf{A}_{\dot{\mathsf{A}}} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{\alpha} \\ \hdashline \mathbf{c}_{a} \\ \hdashline \mathbf{b}_{\dot{\alpha}} \\ \hdashline \mathbf{d}_{\dot{a}} \end{pmatrix}. \tag{2.32}$$

Here we introduced an additional piece of notation that will be used at times. We spelled out the superoscillators $\mathbf{A}_{\mathsf{A}}$ in terms of bosonic oscillators $\mathbf{a}_{\alpha}$ and fermionic $\mathbf{c}_{a}$ with $\alpha = 1, \ldots, p$ and $a = 1, \ldots, r$. In the same way $\mathbf{A}_{\dot{\mathsf{A}}}$ is written using bosonic $\mathbf{b}_{\dot{\alpha}}$ and fermionic $\mathbf{d}_{\dot{a}}$ with $\dot{\alpha} = 1, \ldots, q$ and $\dot{a} = 1, \ldots, s$. Analogous notation applies for the creation operators $\bar{\mathbf{A}}_{\mathcal{A}}$. This terminology is inspired by [104] and [40]. Next, one verifies that the "particle-hole" transformation

$$\begin{pmatrix} \mathbf{A}_{\mathsf{A}} \\ \hdashline \mathbf{A}_{\dot{\mathsf{A}}} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{A}_{\mathsf{A}} \\ \hdashline \bar{\mathbf{A}}_{\dot{\mathsf{A}}} \end{pmatrix}, \quad \begin{pmatrix} \bar{\mathbf{A}}_{\mathsf{A}} \\ \hdashline \bar{\mathbf{A}}_{\dot{\mathsf{A}}} \end{pmatrix} \mapsto \begin{pmatrix} \bar{\mathbf{A}}_{\mathsf{A}} \\ \hdashline -(-1)^{|\dot{\mathsf{A}}|}\mathbf{A}_{\dot{\mathsf{A}}} \end{pmatrix} \tag{2.33}$$

is an automorphism of the superoscillator algebra (2.29). Note, however, that it breaks (2.30). This is essential for the transition from a compact to a non-compact algebra, because otherwise the reality conditions of the $\mathfrak{gl}(n|m)$ generators $\mathbf{J}_{\mathcal{A}\mathcal{B}}$ would not be altered. Applying this automorphism to the generators (2.31) yields[5]

$$(\mathbf{J}_{\mathcal{A}\mathcal{B}}) = \left(\begin{array}{c:c} \mathbf{J}_{\mathsf{A}\mathsf{B}} & \mathbf{J}_{\mathsf{A}\dot{\mathsf{B}}} \\ \hdashline \mathbf{J}_{\dot{\mathsf{A}}\mathsf{B}} & \mathbf{J}_{\dot{\mathsf{A}}\dot{\mathsf{B}}} \end{array}\right) = \left(\begin{array}{c:c} \bar{\mathbf{A}}_{\mathsf{A}}\mathbf{A}_{\mathsf{B}} & \bar{\mathbf{A}}_{\mathsf{A}}\bar{\mathbf{A}}_{\dot{\mathsf{B}}} \\ \hdashline -(-1)^{|\dot{\mathsf{A}}|}\mathbf{A}_{\dot{\mathsf{A}}}\mathbf{A}_{\mathsf{B}} & -(-1)^{|\dot{\mathsf{A}}|}\mathbf{A}_{\dot{\mathsf{A}}}\bar{\mathbf{A}}_{\dot{\mathsf{B}}} \end{array}\right). \tag{2.34}$$

Notice that the blocks $\mathbf{J}_{\mathsf{A}\mathsf{B}}$ and $\mathbf{J}_{\dot{\mathsf{A}}\dot{\mathsf{B}}}$ realize the subalgebras $\mathfrak{gl}(p|r)$ and $\mathfrak{gl}(q|s)$, respectively. Let $\mathcal{D}_c \subset \mathcal{F}$ be the eigenspace of the central element

$$\mathbf{C} = \operatorname{tr}(\mathbf{J}_{\mathcal{A}\mathcal{B}}) = \sum_{\mathsf{A}} \bar{\mathbf{A}}_{\mathsf{A}}\mathbf{A}_{\mathsf{A}} - \sum_{\dot{\mathsf{A}}}(-1)^{|\dot{\mathsf{A}}|}\mathbf{A}_{\dot{\mathsf{A}}}\bar{\mathbf{A}}_{\dot{\mathsf{A}}} \tag{2.35}$$

[5]Compared to [2] we changed the position of the minus signs even in the purely bosonic case.

with eigenvalue c. For each $c \in \mathbb{Z}$ this infinite-dimensional space forms a unitary representation of the algebra $\mathfrak{u}(p,q|r+s)$, see [140]. Hence we may interpret c as a representation label. The space $\mathcal{D}_c$ contains a lowest weight state, which by definition is annihilated by all $\mathbf{J}_{\mathcal{AB}}$ with $\mathcal{A} > \mathcal{B}$, i.e. by the strictly lower triangular entries of the matrix (2.34). Notice that in the special case $q = 0$ or $p = 0$ the space $\mathcal{D}_c$ is finite-dimensional.

According to (2.27), Yangian invariants are in particular $\mathfrak{gl}(n|m)$ singlet states. For such states to exist, we need also spin chain sites with representations that are dual to the class of representations $\mathcal{D}_c$. We define such *dual representations* with generators $\bar{\mathbf{J}}_{\mathcal{AB}} = -(-1)^{|\mathcal{A}|+|\mathcal{A}||\mathcal{B}|}\mathbf{J}^\dagger_{\mathcal{AB}}$ that are obtained from (2.34) employing the automorphism (2.6),

$$(\bar{\mathbf{J}}_{\mathcal{AB}}) = \left(\begin{array}{c:c} \bar{\mathbf{J}}_{\mathsf{AB}} & \bar{\mathbf{J}}_{\mathsf{A\dot{B}}} \\ \hdashline \bar{\mathbf{J}}_{\mathsf{\dot{A}B}} & \bar{\mathbf{J}}_{\mathsf{\dot{A}\dot{B}}} \end{array}\right) = \left(\begin{array}{c:c} -(-1)^{|\mathsf{A}|+|\mathsf{A}||\mathsf{B}|}\bar{\mathbf{A}}_{\mathsf{B}}\mathbf{A}_{\mathsf{A}} & -(-1)^{|\mathsf{A}|+|\mathsf{\dot{B}}|+|\mathsf{A}||\mathsf{\dot{B}}|}\mathbf{A}_{\mathsf{\dot{B}}}\mathbf{A}_{\mathsf{A}} \\ \hdashline (-1)^{|\mathsf{\dot{A}}|+|\mathsf{\dot{A}}||\mathsf{B}|}\bar{\mathbf{A}}_{\mathsf{B}}\bar{\mathbf{A}}_{\mathsf{\dot{A}}} & (-1)^{|\mathsf{\dot{A}}|+|\mathsf{\dot{B}}|+|\mathsf{\dot{A}}||\mathsf{\dot{B}}|}\mathbf{A}_{\mathsf{\dot{B}}}\bar{\mathbf{A}}_{\mathsf{\dot{A}}} \end{array}\right). \tag{2.36}$$

We denote by $\bar{\mathcal{D}}_c \subset \mathcal{F}$ the eigenspace of the central element

$$\bar{\mathbf{C}} = \operatorname{tr}(\bar{\mathbf{J}}_{\mathcal{AB}}) = -\sum_{\mathsf{A}}\bar{\mathbf{A}}_{\mathsf{A}}\mathbf{A}_{\mathsf{A}} + \sum_{\mathsf{\dot{A}}}(-1)^{|\mathsf{\dot{A}}|}\mathbf{A}_{\mathsf{\dot{A}}}\bar{\mathbf{A}}_{\mathsf{\dot{A}}} \tag{2.37}$$

with eigenvalue c. For each $c \in \mathbb{Z}$ this space carries a unitary representation of $\mathfrak{u}(p,q|r+s)$. The representation $\bar{\mathcal{D}}_c$ is dual to $\mathcal{D}_{-c}$. It contains a highest weight state, which is annihilated by all $\bar{\mathbf{J}}_{\mathcal{AB}}$ with $\mathcal{A} < \mathcal{B}$. In case of $q = 0$ or $p = 0$ the space $\bar{\mathcal{D}}_c$ is finite dimensional. Notice that for the compact generators (2.31) the automorphism (2.6) does agree with (2.7). However, for the non-compact generators (2.34) it does not.

Having defined the two classes of oscillator representations allows us to use them at the sites of the monodromy $M(u)$ in (2.19). At each site we chose either an "ordinary" representation $\mathcal{D}_{c_i}$ with generators $J^i_{\mathcal{AB}} = \mathbf{J}^i_{\mathcal{AB}}$ or a "dual" representation $\bar{\mathcal{D}}_{c_i}$ with $J^i_{\mathcal{AB}} = \bar{\mathbf{J}}^i_{\mathcal{AB}}$. The monodromy $M(u)$, and hence the representation of the Yangian, is completely specified by $2N$ parameters, i.e. N inhomogeneities $v_i \in \mathbb{C}$ and N representation labels $c_i \in \mathbb{Z}$. We remark that the tensor product decomposition of the oscillator representations employed at the spin chain sites was studied in [141] for the $\mathfrak{u}(p,q)$ case, see also e.g. [142, 143] for exemplary results.

Let us add a comment on the necessity of the two classes of representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_c$. In section 1.3.4 the $\mathfrak{gl}(4|4)$ generators (1.30), which are expressed in terms of the spinors λ and $\tilde{\lambda}$, seem to look alike at all N sites of the amplitude. However, taking into account the reality condition (1.5) for the spinors, $\tilde{\lambda} = \pm\bar{\lambda}$, there are two different kinds of generators. In section 4.2 we will show that these correspond to the two classes of representations introduced here.

2.4 Sample Invariants

2.4.1 Compact Bosonic Invariants

We have the Yangian algebra and oscillator representations of the non-compact superalgebra $\mathfrak{u}(p,q|m)$ at our disposal. Therefore we could in principle start looking for solutions $|\Psi\rangle$ of the Yangian invariance condition (2.24) for these representations. However, to gain a better understanding of the oscillator formalism, we choose to concentrate on the special case of the compact bosonic algebra $\mathfrak{u}(n) \subset \mathfrak{gl}(n)$ for the time being. We present some details on the oscillator representations and the Lax operators in this case in section 2.4.1.1.

This is followed in section 2.4.1.2 by a reformulation of the Yangian invariance condition as an intertwining relation and thereby emphasizing its interpretation as a Yang-Baxter-like equation. The remaining sections contain sample Yangian invariants. The structure of these invariants is relatively simple. They are polynomials in the creation operators acting on the Fock vacuum because the representations we are dealing with are finite-dimensional. We will encounter these compact bosonic sample invariants again in section 3.2 where we construct them using a Bethe ansatz.

Let us remark that these sample invariants with finite-dimensional $\mathfrak{u}(n)$ representations can be brought into a form which makes them look very much akin to tree-level $\mathcal{N}=4$ SYM amplitudes with infinite-dimensional $\mathfrak{psu}(2,2|4)$ representations, see [1]. This reformulation involves the Graßmannian integral formulation of the amplitudes in terms of supertwistors, which is similar to that with spinor helicity variables reviewed in section 1.3.5. By the nature of the differing representations involved the argument is necessarily somewhat formal. Nevertheless, it provides further motivation for the study of compact bosonic sample invariants.

2.4.1.1 Details on Representations and Lax Operators

After introducing the classes of oscillator representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_c$ of the non-compact superalgebra $\mathfrak{u}(p,q|r+s) \subset \mathfrak{gl}(n|m)$ in section 2.3, we present additional details on these representations for the special case $q=r=s=0$. Thus we are concentrating on unitary representations of $\mathfrak{u}(n) \subset \mathfrak{gl}(n)$. In this case the $\mathfrak{gl}(n)$ generators (2.34) associated with the class $\mathcal{D}_c$ and (2.36) of $\bar{\mathcal{D}}_c$ reduce to, respectively,

$$\mathbf{J}_{\alpha\beta} = \bar{\mathbf{a}}_\alpha \mathbf{a}_\beta\,, \quad \bar{\mathbf{J}}_{\alpha\beta} = -\bar{\mathbf{a}}_\beta \mathbf{a}_\alpha \tag{2.38}$$

with $\alpha, \beta = 1, \ldots, n$. Here we used (2.32) to express the generators in terms of *bosonic oscillators*,

$$[\mathbf{a}_\alpha, \bar{\mathbf{a}}_\beta] = \delta_{\alpha\beta}\,, \quad [\mathbf{a}_\alpha, \mathbf{a}_\beta] = 0\,, \quad [\bar{\mathbf{a}}_\alpha, \bar{\mathbf{a}}_\beta] = 0\,, \quad \mathbf{a}_\alpha^\dagger = \bar{\mathbf{a}}_\alpha\,, \quad \mathbf{a}_\alpha|0\rangle = 0\,, \tag{2.39}$$

where the brackets denote the commutator. A review of these realizations of the $\mathfrak{gl}(n)$ algebra, which are sometimes said to be of Jordan-Schwinger-type, may be found e.g. in [144]. The central elements (2.35) and (2.37) become, up to a sign, number operators,

$$\mathbf{C} = \sum_{\alpha=1}^{n} \bar{\mathbf{a}}_\alpha \mathbf{a}_\alpha\,, \quad \bar{\mathbf{C}} = -\sum_{\alpha=1}^{n} \bar{\mathbf{a}}_\alpha \mathbf{a}_\alpha\,. \tag{2.40}$$

In the non-compact case their eigenvalues c can be arbitrary integers. From (2.40) we conclude that in the compact case for the class $\mathcal{D}_c$ we have $c \in \mathbb{N}$, while for $\bar{\mathcal{D}}_c$ one needs $c \in -\mathbb{N}$. Hence the representation spaces $\mathcal{D}_c$ and $\bar{\mathcal{D}}_{-c}$ with $c \in \mathbb{N}$ are the finite-dimensional subspace of the Fock space $\mathcal{F}$ consisting of polynomials of degree c in the creation operators $\bar{\mathbf{a}}_\alpha$ acting on $|0\rangle$. Both representations posses a *highest weight state*,

$$|\sigma\rangle = (\bar{\mathbf{a}}_1)^c|0\rangle \in \mathcal{D}_c\,, \quad |\bar{\sigma}\rangle = (\bar{\mathbf{a}}_n)^c|0\rangle \in \bar{\mathcal{D}}_{-c}\,. \tag{2.41}$$

These states are characterized by

$$\begin{aligned} &\mathbf{J}_{\alpha\beta}|\sigma\rangle = 0 \quad \text{for} \quad \alpha < \beta\,, && \bar{\mathbf{J}}_{\alpha\beta}|\bar{\sigma}\rangle = 0 \quad \text{for} \quad \alpha < \beta\,, \\ &\mathbf{J}_{\alpha\alpha}|\sigma\rangle = c\,\delta_{1\,\alpha}|\sigma\rangle\,, && \bar{\mathbf{J}}_{\alpha\alpha}|\bar{\sigma}\rangle = -c\,\delta_{n\,\alpha}|\bar{\sigma}\rangle\,. \end{aligned} \tag{2.42}$$

Because both representations are finite-dimensional, they also contain a lowest weight state, that we do not state here. The $\mathfrak{gl}(n)$ weight $\Xi = (\xi^{(1)}, \ldots, \xi^{(n)})$ of a state $|\phi\rangle$ is defined by

$J_{\alpha\alpha}|\phi\rangle = \xi^{(\alpha)}|\phi\rangle$. Therefore, from (2.42) we read off the highest weights of the two classes of oscillator representations,

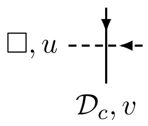
$$\Xi = (c, 0, \ldots, 0) \quad \text{for} \quad \mathcal{D}_c\,, \quad \Xi = (0, \ldots, 0, -c) \quad \text{for} \quad \bar{\mathcal{D}}_{-c}\,. \tag{2.43}$$

From its highest weight we may identify $\mathcal{D}_c$ with the class of totally symmetric representations of $\mathfrak{gl}(n)$. Notice that in the non-compact setting only $\bar{\mathcal{D}}_{-c}$ has a highest weight state, while $\mathcal{D}_c$ contains merely a lowest weight state. Knowing that in the compact case both types of oscillators representations have a highest weight will be of great importance in section 3.2. The Bethe ansatz employed there for the construction of Yangian invariants makes crucial use of highest weight states. For completeness, we recall from section 2.3 that $\bar{\mathcal{D}}_{-c}$ is the *dual representation* to $\mathcal{D}_c$ is the sense that the generators (2.38) satisfy

$$\bar{\mathbf{J}}_{\alpha\beta} = -\mathbf{J}^\dagger_{\alpha\beta}\,. \tag{2.44}$$

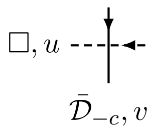
Next, we provide some details on the Lax operators (2.18) for the compact bosonic oscillator representations. With (2.38) they read

$$R_{\square\,\mathcal{D}_c}(u-v) = f_{\mathcal{D}_c}(u-v)\left(1 + (u-v)^{-1}\sum_{\alpha,\beta=1}^{n} E_{\alpha\beta}\bar{\mathbf{a}}_\beta\mathbf{a}_\alpha\right) = \tag{2.45}$$



,

$$R_{\square\,\bar{\mathcal{D}}_{-c}}(u-v) = f_{\bar{\mathcal{D}}_{-c}}(u-v)\left(1 - (u-v)^{-1}\sum_{\alpha,\beta=1}^{n} E_{\alpha\beta}\bar{\mathbf{a}}_\alpha\mathbf{a}_\beta\right) = \tag{2.46}$$



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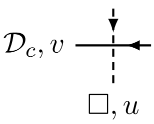
We severely constrain the normalizations $f_{\bar{\mathcal{D}}_{-c}}(u)$ and $f_{\mathcal{D}_c}(u)$ by demanding that each of the Lax operators satisfies two additional equations, a unitarity and a crossing relation. The *unitarity relations* read

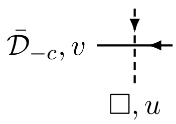
$$R_{\square\,\mathcal{D}_c}(u-v)R_{\mathcal{D}_c\,\square}(v-u) = 1\,, \quad R_{\square\,\bar{\mathcal{D}}_{-c}}(u-v)R_{\bar{\mathcal{D}}_{-c}\,\square}(v-u) = 1\,. \tag{2.47}$$

They contain the two further Lax operators

$$R_{\mathcal{D}_c\,\square}(v-u) = \tag{2.48}$$



, $R_{\bar{\mathcal{D}}_{-c}\,\square}(v-u) =$



,

where the order of auxiliary and quantum spaces is exchanged. These Lax operators solve the Yang-Baxter equation in $\square\otimes\mathcal{D}_c\otimes\square$ and $\square\otimes\bar{\mathcal{D}}_{-c}\otimes\square$, respectively. In these equations they are the only unknowns, cf. (2.11). The Lax operators (2.48) can be written in terms of those in (2.45) and (2.46),

$$\begin{aligned} R_{\mathcal{D}_c\,\square}(v-u) &= R_{\square\,\mathcal{D}_c}(v-u-c+1)\,, \\ R_{\bar{\mathcal{D}}_{-c}\,\square}(v-u) &= R_{\square\,\bar{\mathcal{D}}_{-c}}(v-u+n+c-1)\,. \end{aligned} \tag{2.49}$$

In this sense they are symmetric up to a shift of the spectral parameter. Form the unitarity conditions in (2.47) we obtain constraints on the normalization of the Lax operators,

$$f_{\mathcal{D}_c}(u)f_{\mathcal{D}_c}(-u-c+1) = \frac{u(u+c-1)}{u(u+c-1)-c}\,, \quad f_{\bar{\mathcal{D}}_{-c}}(u)f_{\bar{\mathcal{D}}_{-c}}(-u+c-1+n) = 1\,. \tag{2.50}$$

The *crossing relations* that we demand are

$$R_{\square\,\bar{\mathcal{D}}_{-c}}(u+\kappa_{\mathcal{D}_c}) = R_{\mathcal{D}_c\,\square}(-u)^\dagger\,, \quad R_{\square\,\mathcal{D}_c}(u+\kappa_{\bar{\mathcal{D}}_{-c}}) = R_{\bar{\mathcal{D}}_{-c}\,\square}(-u)^\dagger\,, \tag{2.51}$$

where $\kappa_{\mathcal{D}_c}$ and $\kappa_{\bar{\mathcal{D}}_{-c}}$ are called crossing parameters and the conjugation only acts on the oscillators. These equations impose

$$\kappa_{\mathcal{D}_c} = c-1\,, \quad \kappa_{\bar{\mathcal{D}}_{-c}} = -c+1-n\,, \quad f_{\bar{\mathcal{D}}_{-c}}(u) = f_{\mathcal{D}_c}(-u)\,. \tag{2.52}$$

Note that the two equations in (2.51) lead only to one condition on the normalizations. A solution of (2.50) and (2.52) is given by the well-known formula

$$f_{\mathcal{D}_c}(u) = \frac{\Gamma\big(\frac{1-u}{n}\big)\Gamma\big(\frac{n+u}{n}\big)}{\Gamma\big(\frac{1-c-u}{n}\big)\Gamma\big(\frac{n+c+u}{n}\big)}\,. \tag{2.53}$$

In case of $c=1$ it was obtained in [145]. For higher integer values of c it can be constructed by means of the recursion relation

$$f_{\mathcal{D}_c}(u) f_{\mathcal{D}_{c'}}(u+c) = f_{\mathcal{D}_{c+c'}}(u)\,. \tag{2.54}$$

Let us remark that the solution (2.53) is not unique, see once again [145].

2.4.1.2 Yangian Invariance as Yang-Baxter Equation

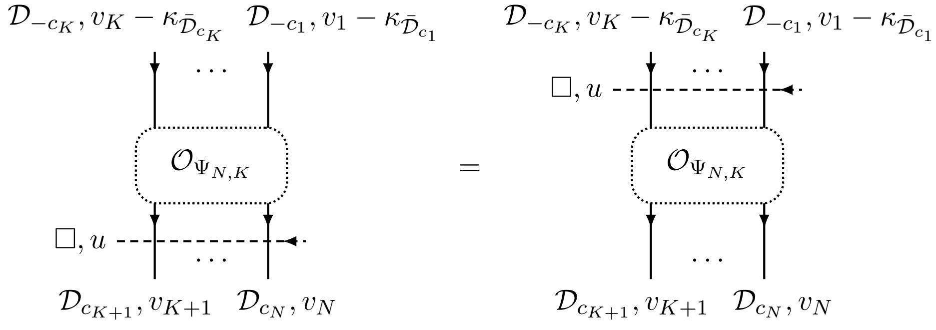
Here we present a reformulation of the Yangian invariance condition (2.24) as a Yang-Baxter-like equation. By way of example, we consider a monodromy (2.13) where the first K sites of the total quantum space carry a dual representation of the class $\bar{\mathcal{D}}_c$ and the remaining $N-K$ sites carry an ordinary one of the type $\mathcal{D}_c$. We denote such a monodromy by $M_{N,K}(u)$ and an associated solution of (2.24) will be labeled $|\Psi_{N,K}\rangle$. This is motivated by the notation employed for the amplitudes $\mathcal{A}_{N,K}^{(\text{tree})}$ in section 1.3. The monodromy reads

$$\begin{aligned} M_{N,K}(u) =& R_{\square\,\bar{\mathcal{D}}_{c_1}}(u-v_1)\cdots R_{\square\,\bar{\mathcal{D}}_{c_K}}(u-v_K) \\ &\cdot R_{\square\,\mathcal{D}_{c_{K+1}}}(u-v_{K+1})\cdots R_{\square\,\mathcal{D}_{c_N}}(u-v_N)\,. \end{aligned} \tag{2.55}$$

Because we are in the compact case the representation labels satisfy $c_1,\ldots,c_K \in -\mathbb{N}$ and $c_{K+1},\ldots,c_N \in \mathbb{N}$. The Yangian invariance condition (2.24) can be interpreted as an intertwining relation of the tensor product of the first K with the remaining $N-K$ spaces of the total quantum space. To obtain this relation we conjugate (2.24) in the first K spaces and use (2.47) and (2.51) for these spaces. This yields

$$\begin{aligned} &R_{\square\,\mathcal{D}_{c_{K+1}}}(u-v_{K+1})\cdots R_{\square\,\mathcal{D}_{c_N}}(u-v_N)\mathcal{O}_{\Psi_{N,K}} \\ &= \mathcal{O}_{\Psi_{N,K}} R_{\square\,\mathcal{D}_{-c_K}}(u-v_K+\kappa_{\bar{\mathcal{D}}_{c_K}})\cdots R_{\square\,\mathcal{D}_{-c_1}}(u-v_1+\kappa_{\bar{\mathcal{D}}_{c_1}})\,, \end{aligned} \tag{2.56}$$

where $\mathcal{O}_{\Psi_{N,K}} := |\Psi_{N,K}\rangle^{\dagger_1\cdots\dagger_K}$. It is represented graphically as



. (2.57)

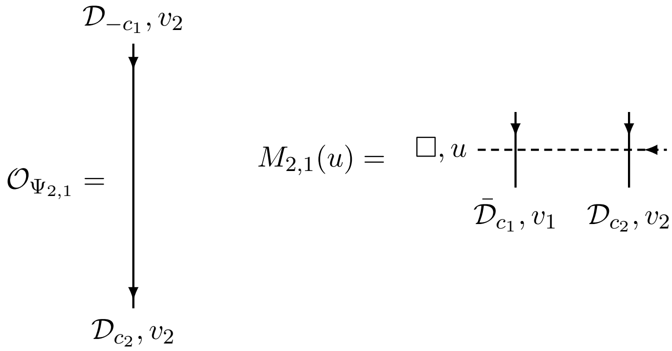


Figure 2.1: The left side shows a graphical depiction of the intertwiner version $\mathcal{O}_{\Psi_{2,1}}$ found in (2.63) of the Yangian invariant vector $|\Psi_{2,1}\rangle$. This intertwiner is essentially an identity operator and is thus represented by a line. On the right we display the monodromy $M_{2,1}(u)$ belonging to the Yangian invariant. The representation labels and inhomogeneities of the intertwiner as well as the monodromy obey (2.59) to ensure Yangian invariance.

The analogy of this equation with the Yang-Baxter equation will allow us to give a graphical interpretation of some sample intertwiners $\mathcal{O}_{\Psi_{N,K}}$ similar to the one for the R-matrix in (2.10). An equation like (2.56) was formulated in [104] in the context of deformed amplitudes of $\mathcal{N} = 4$ SYM.

2.4.1.3 Two-Site Invariant and Identity Operator

At this point everything is set up to construct the first and simplest sample solution of the Yangian invariance condition (2.24). We consider a monodromy of the form (2.55) with $N = 2$ sites out of which $K = 1$ are dual,

$$M_{2,1}(u) = R_{\square\,\bar{\mathcal{D}}_{c_1}}(u - v_1) R_{\square\,\mathcal{D}_{c_2}}(u - v_2)\,, \tag{2.58}$$

see also figure 2.1. The Lax operators of this monodromy are given explicitly in (2.45) and (2.46). Recall from (2.27) that the Yangian invariant $|\Psi_{2,1}\rangle$ we want to construct is in particular a $\mathfrak{gl}(n)$ singlet. Hence we pick a monodromy where one site carries an ordinary representation $\mathcal{D}_{c_2}$ and the other one a dual representation $\bar{\mathcal{D}}_{c_1}$. In trying to solve (2.24) with this monodromy we find that a solution only exists if the inhomogeneities and representation labels obey

$$v_1 = v_2 - n - c_2 + 1\,, \quad c_1 + c_2 = 0\,. \tag{2.59}$$

The latter equation was to be expected because it guarantees that the two representations are in fact dual to each other, cf. the discussion around (2.44). The normalization of the monodromy (2.58) derives from those of the Lax operators in (2.45) and (2.46). It turns out to be trivial,

$$f_{\bar{\mathcal{D}}_{c_1}}(u - v_1) f_{\mathcal{D}_{c_2}}(u - v_2) = 1\,. \tag{2.60}$$

This is shown using (2.59), the unitarity condition for $f_{\bar{\mathcal{D}}_{-c}}(u)$ in (2.50) and the relation between the normalizations $f_{\mathcal{D}_c}(u)$ and $f_{\bar{\mathcal{D}}_{-c}}(u)$ in (2.52). At this stage the solution of

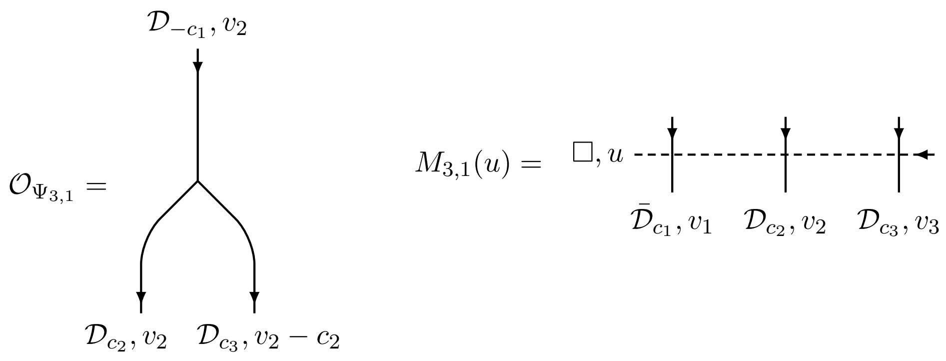


Figure 2.2: The left part represents the intertwiner $\mathcal{O}_{\Psi_{3,1}}$ that we computed in (2.69). It is represented by a trivalent vertex. To its right we depict the monodromy $M_{3,1}(u)$ associated with the Yangian invariant $|\Psi_{3,1}\rangle$. The necessary constraints on the representation labels and inhomogeneities may be found in (2.65).

(2.24) is easily shown to be

$$|\Psi_{2,1}\rangle = (1 \bullet 2)^{c_2}|0\rangle \quad \text{with} \quad (l \bullet k) = \sum_{\alpha=1}^{n} \bar{\mathbf{a}}^k_\alpha \bar{\mathbf{a}}^l_\alpha \,. \tag{2.61}$$

Here the upper indices on the oscillators indicate the site of the monodromy. The invariant $|\Psi_{2,1}\rangle$ is unique up to a scalar prefactor, which evidently drops out of (2.24).

The intertwiner belonging to the invariant $|\Psi_{2,1}\rangle$ is obtained from (2.56) with $K = 1$ and using $\kappa_{\bar{\mathcal{D}}_{c_1}}$ given in (2.52). We are led to

$$R_{\square\,\mathcal{D}_{c_2}}(u - v_2)\mathcal{O}_{\Psi_{2,1}} = \mathcal{O}_{\Psi_{2,1}} R_{\square\,\mathcal{D}_{-c_1}}(u - v_2) \tag{2.62}$$

with

$$\mathcal{O}_{\Psi_{2,1}} := |\Psi_{2,1}\rangle^{\dagger_1} = \sum_{\alpha_1,\ldots,\alpha_{c_2}=1}^{n} \bar{\mathbf{a}}^2_{\alpha_1} \cdots \bar{\mathbf{a}}^2_{\alpha_{c_2}} |0\rangle\langle 0| \mathbf{a}^1_{\alpha_1} \cdots \mathbf{a}^1_{\alpha_{c_2}} \,. \tag{2.63}$$

The intertwiner $\mathcal{O}_{\Psi_{2,1}}$ may be understood as $c_2!$ times the identity operator because with $c_1 = -c_2$ in (2.59) the representation spaces $\mathcal{D}_{-c_1}$ and $\mathcal{D}_{c_2}$ can be identified. Hence we represent $\mathcal{O}_{\Psi_{2,1}}$ graphically in figure 2.1 by a line. In analogy we may also think of the "black box" for the invariant $|\Psi_{2,1}\rangle$ in the graphical version (2.26) of the Yangian invariance condition as a line.

2.4.1.4 Three-Site Invariants and Bootstrap Equations

Apart from the two-site Yangian invariant $|\Psi_{2,1}\rangle$, the next simplest invariants are $|\Psi_{3,1}\rangle$ or $|\Psi_{3,2}\rangle$ with three sites. These are characterized by three-site monodromies, where also here we place the sites carrying a dual representation left of those with an ordinary one. A graphical interpretation of the three-site invariants $|\Psi_{3,1}\rangle$ and $|\Psi_{3,2}\rangle$ as trivalent vertices is provided in figures 2.2 and 2.3, respectively.

The said monodromy with one dual site characterizing $|\Psi_{3,1}\rangle$ is

$$M_{3,1}(u) = R_{\square\,\bar{\mathcal{D}}_{c_1}}(u - v_1) R_{\square\,\mathcal{D}_{c_2}}(u - v_2) R_{\square\,\mathcal{D}_{c_3}}(u - v_3) \,, \tag{2.64}$$

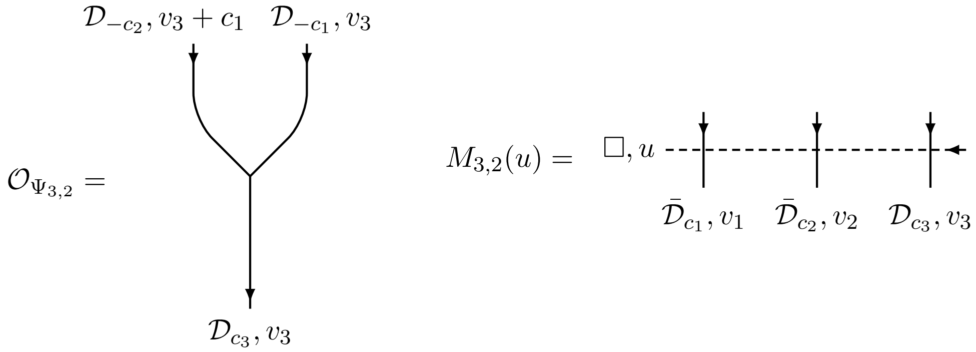


Figure 2.3: The intertwiner $\mathcal{O}_{\Psi_{3,2}}$ from (2.75) is visualized on the left side as a trivalent vertex. The monodromy $M_{3,2}(u)$ of the corresponding Yangian invariant $|\Psi_{3,2}\rangle$ can be found on the right side. The parameters of the intertwiner and the monodromy obey the constraints (2.71).

cf. the left site of figure 2.2. In order to find a solution of the Yangian invariance condition (2.24), we choose the parameters

$$v_2 = v_1 + n + c_2 + c_3 - 1\,, \quad v_3 = v_1 + n + c_3 - 1\,, \quad c_1 + c_2 + c_3 = 0\,. \tag{2.65}$$

Taking into account this choice the normalization of the monodromy (2.64) with Lax operators of the types (2.45) and (2.46) trivializes,

$$f_{\bar{\mathcal{D}}_{c_1}}(u - v_1) f_{\mathcal{D}_{c_2}}(u - v_2) f_{\mathcal{D}_{c_3}}(u - v_3) = 1\,. \tag{2.66}$$

To show this we used (2.54) for $f_{\mathcal{D}_c}(u)$, the unitarity condition for $f_{\bar{\mathcal{D}}_{-c}}(u)$ and we expressed $f_{\bar{\mathcal{D}}_{-c}}(u)$ in terms of $f_{\mathcal{D}_c}(u)$ by means of (2.52). Then by a straightforward computation one verifies that the Yangian invariant solving (2.24) is

$$|\Psi_{3,1}\rangle = (1 \bullet 2)^{c_2} (1 \bullet 3)^{c_3} |0\rangle\,, \tag{2.67}$$

where we fixed a scalar prefactor.

We continue with the discussion of the corresponding intertwining relation. It is obtained from the general relation (2.56) by restricting to $K = 1$ and using $\kappa_{\bar{\mathcal{D}}_{c_1}}$ from (2.52),

$$R_{\square\,\mathcal{D}_{c_2}}(u - v_2) R_{\square\,\mathcal{D}_{c_3}}(u - v_2 + c_2) \mathcal{O}_{\Psi_{3,1}} = \mathcal{O}_{\Psi_{3,1}} R_{\square\,\mathcal{D}_{-c_1}}(u - v_2) \tag{2.68}$$

with

$$\mathcal{O}_{\Psi_{3,1}} := |\Psi_{3,1}\rangle^{\dagger_1} = \sum_{\substack{\alpha_1,\ldots,\alpha_{c_2}\\ \beta_1,\ldots,\beta_{c_3}}} \bar{\mathbf{a}}^2_{\alpha_1} \cdots \bar{\mathbf{a}}^2_{\alpha_{c_2}} \bar{\mathbf{a}}^3_{\beta_1} \cdots \bar{\mathbf{a}}^3_{\beta_{c_3}} |0\rangle\langle 0| \mathbf{a}^1_{\alpha_1} \cdots \mathbf{a}^1_{\alpha_{c_2}} \mathbf{a}^1_{\beta_1} \cdots \mathbf{a}^1_{\beta_{c_3}}\,. \tag{2.69}$$

Such an intertwining relation is known as bootstrap equation [105], see also e.g. [106, 107].

Next we investigate the monodromy

$$M_{3,2}(u) = R_{\square\,\bar{\mathcal{D}}_{c_1}}(u - v_1) R_{\square\,\bar{\mathcal{D}}_{c_2}}(u - v_2) R_{\square\,\mathcal{D}_{c_3}}(u - v_3) \tag{2.70}$$

with two dual sites on the left. It is depicted on the right side of figure 2.3. In order to find the solution $|\Psi_{3,2}\rangle$ of (2.24) with this monodromy, we demand

$$v_1 = v_3 - n + c_1 + 1\,, \quad v_2 = v_3 - n - c_3 + 1\,, \quad c_1 + c_2 + c_3 = 0\,. \tag{2.71}$$

Analogous to the case of the other three-site invariant, these conditions ensure that the normalization of the monodromy (2.70) trivializes,

$$f_{\bar{\mathcal{D}}_{c_1}}(u - v_1) f_{\bar{\mathcal{D}}_{c_2}}(u - v_2) f_{\mathcal{D}_{c_3}}(u - v_3) = 1\,. \tag{2.72}$$

We solve (2.24) by an explicit calculation that yields

$$|\Psi_{3,2}\rangle = (1 \bullet 3)^{-c_1}(2 \bullet 3)^{-c_2}|0\rangle\,, \tag{2.73}$$

where once more we picked a scalar prefactor. Recall that $c_1, c_2 \in -\mathbb{N}$ for the compact representations under consideration here.

The intertwiner version of the Yangian invariance condition is obtained from (2.56) by setting $K = 2$ and using the values of $\kappa_{\bar{\mathcal{D}}_{c_1}}, \kappa_{\bar{\mathcal{D}}_{c_2}}$ from (2.52). This results in the bootstrap equation

$$R_{\square\,\mathcal{D}_{c_3}}(u - v_3)\mathcal{O}_{\Psi_{3,2}} = \mathcal{O}_{\Psi_{3,2}} R_{\square\,\mathcal{D}_{-c_2}}(u - v_3 - c_1) R_{\square\,\mathcal{D}_{-c_1}}(u - v_3) \tag{2.74}$$

with the solution

$$\mathcal{O}_{\Psi_{3,2}} := |\Psi_{3,2}\rangle^{\dagger_1\dagger_2} = \sum_{\substack{\alpha_1,\ldots,\alpha_{-c_1}\\ \beta_1,\ldots,\beta_{-c_2}}} \bar{\mathbf{a}}^3_{\alpha_1} \cdots \bar{\mathbf{a}}^3_{\alpha_{-c_1}} \bar{\mathbf{a}}^3_{\beta_1} \cdots \bar{\mathbf{a}}^3_{\beta_{-c_2}} |0\rangle\langle 0| \mathbf{a}^1_{\alpha_1} \cdots \mathbf{a}^1_{\alpha_{-c_1}} \mathbf{a}^2_{\beta_1} \cdots \mathbf{a}^2_{\beta_{-c_2}}\,. \tag{2.75}$$

2.4.1.5 Four-Site Invariant and Yang-Baxter Equation

In this section we address Yangian invariants with four sites. Importantly, we will show that a particular four-site invariant is nothing but the well-known $\mathfrak{gl}(n)$ invariant R-matrix [146] "in disguise". This R-matrix is a solution of the Yang-Baxter equation. For $K = 2$ dual sites in the monodromy (2.55), the intertwining relation (2.56) reduces to this Yang-Baxter equation. Therefore we study the monodromy

$$M_{4,2}(u) = R_{\square\,\bar{\mathcal{D}}_{c_1}}(u - v_1) R_{\square\,\bar{\mathcal{D}}_{c_2}}(u - v_2) R_{\square\,\mathcal{D}_{c_3}}(u - v_3) R_{\square\,\mathcal{D}_{c_4}}(u - v_4)\,, \tag{2.76}$$

see also figure 2.4. For the inhomogeneities and representation labels we choose

$$v_1 = v_3 - n - c_3 + 1\,, \quad v_2 = v_4 - n - c_4 + 1\,, \quad c_1 + c_3 = 0\,, \quad c_2 + c_4 = 0\,. \tag{2.77}$$

The constraints on the representation labels are needed to indeed obtain the Yang-Baxter equation from (2.56). Notice that (2.77) consists of two sets of the conditions we used for the two-site invariant in (2.59). Hence the normalization trivializes similar as in section 2.4.1.3,

$$f_{\bar{\mathcal{D}}_{c_1}}(u - v_1) f_{\bar{\mathcal{D}}_{c_2}}(u - v_2) f_{\mathcal{D}_{c_3}}(u - v_3) f_{\mathcal{D}_{c_4}}(u - v_4) = 1\,. \tag{2.78}$$

To solve the Yangian invariance condition (2.24) we start from the ansatz

$$|\Psi_{4,2}(v_3 - v_4)\rangle := |\Psi_{4,2}\rangle = \sum_{k=0}^{\min(c_3,c_4)} d_k(v_3 - v_4)|\Upsilon_k\rangle \tag{2.79}$$

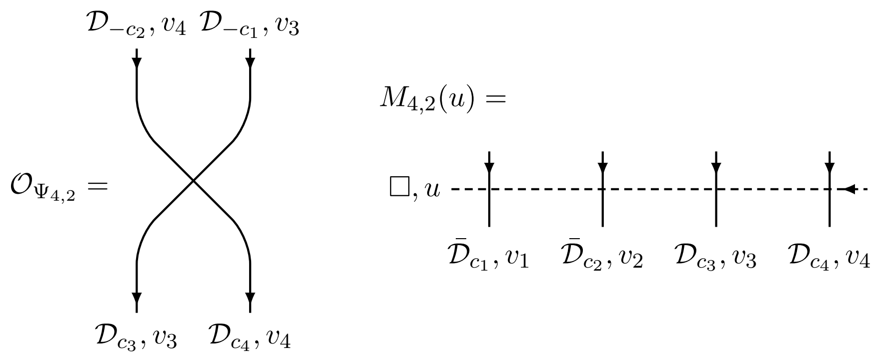


Figure 2.4: The left side shows the intertwiner $\mathcal{O}_{\Psi_{4,2}}$ given in (2.85). We represent it by the intersection of two lines because it can be understood as an R-matrix. The monodromy $M_{4,2}(u)$ of the corresponding Yangian invariant $|\Psi_{4,2}\rangle$ is displayed on the right side. The necessary identifications of the representation labels and the inhomogeneities are written in (2.77).

with the $\mathfrak{gl}(n)$ invariant vectors

$$|\Upsilon_k\rangle = (1 \bullet 3)^{c_3-k}(2 \bullet 4)^{c_4-k}(2 \bullet 3)^k(1 \bullet 4)^k|0\rangle\,. \tag{2.80}$$

It turns out that the four-site invariant depends on a free complex spectral parameter which is the difference of two inhomogeneities,

$$z := v_3 - v_4. \tag{2.81}$$

This dependence is made explicit by the notation $|\Psi_{4,2}(z)\rangle$. The coefficients d_k obey a recursion relation that is obtained from inserting the ansatz (2.79) into (2.24),

$$\frac{d_k(z)}{d_{k+1}(z)} = \frac{(k+1)(z-c_3+k+1)}{(c_3-k)(c_4-k)}\,. \tag{2.82}$$

Its solution is, up to multiplication by a function of k with period 1,

$$d_k(z) = \frac{1}{(c_3-k)!(c_4-k)!k!^2}\,\frac{k!}{\Gamma(z-c_3+k+1)}\,. \tag{2.83}$$

The intertwiner corresponding to $|\Psi_{4,2}(z)\rangle$ solves (2.56) with $K=2$ and $\kappa_{\bar{\mathcal{D}}_{c_1}}$, $\kappa_{\bar{\mathcal{D}}_{c_2}}$ from (2.52). This equation has the form of a Yang-Baxter equation,

$$R_{\square\,\mathcal{D}_{c_3}}(u-v_3)R_{\square\,\mathcal{D}_{c_4}}(u-v_4)\mathcal{O}_{\Psi_{4,2}(z)} = \mathcal{O}_{\Psi_{4,2}(z)}R_{\square\,\mathcal{D}_{-c_2}}(u-v_4)R_{\square\,\mathcal{D}_{-c_1}}(u-v_3)\,, \tag{2.84}$$

where

$$\mathcal{O}_{\Psi_{4,2}(z)} := |\Psi_{4,2}(z)\rangle^{\dagger_1\dagger_2} = \sum_{k=0}^{\min(s_3,s_4)} d_k(z)\mathcal{O}_{\Upsilon_k}\,, \tag{2.85}$$

and

$$\begin{aligned}\mathcal{O}_{\Upsilon_k} := |\Upsilon_k\rangle^{\dagger_1\dagger_2} = \sum_{\substack{\alpha_1,\ldots,\alpha_{c_3}\\ \beta_1,\ldots,\beta_{c_4}}} &\bar{\mathbf{a}}^3_{\alpha_1}\cdots\bar{\mathbf{a}}^3_{\alpha_{c_3}}\bar{\mathbf{a}}^4_{\beta_1}\cdots\bar{\mathbf{a}}^4_{\beta_{c_4}}|0\rangle\\ &\cdot\langle 0|\mathbf{a}^1_{\alpha_1}\cdots\mathbf{a}^1_{\alpha_{c_3-k}}\mathbf{a}^1_{\beta_{c_4-k+1}}\cdots\mathbf{a}^1_{\beta_{c_4}}\\ &\cdot\mathbf{a}^2_{\beta_1}\cdots\mathbf{a}^2_{\beta_{c_4-k}}\mathbf{a}^2_{\alpha_{c_3-k+1}}\cdots\mathbf{a}^2_{\alpha_{c_3}}\,.\end{aligned} \tag{2.86}$$

To obtain a more standard formulation of the Yang-Baxter equation, we identify space the $\mathcal{D}_{-c_1}$ with $\mathcal{D}_{c_3}$ and $\mathcal{D}_{-c_2}$ with $\mathcal{D}_{c_4}$ and rename $\mathcal{O}_{\Psi_{4,2}(z)}$ as $R_{\mathcal{D}_{c_3}\mathcal{D}_{c_4}}(z)$. Then (2.86) translates into

$$R_{\square\,\mathcal{D}_{c_3}}(u-v_3)R_{\square\,\mathcal{D}_{c_4}}(u-v_4)R_{\mathcal{D}_{c_3}\mathcal{D}_{c_4}}(z) = R_{\mathcal{D}_{c_3}\mathcal{D}_{c_4}}(z)R_{\square\,\mathcal{D}_{c_4}}(u-v_4)R_{\square\,\mathcal{D}_{c_3}}(u-v_3)\,. \tag{2.87}$$

Consequently $R_{\mathcal{D}_{c_3}\mathcal{D}_{c_4}}(z)$ is the $\mathfrak{gl}(n)$ invariant R-matrix of [146] for symmetric representations, which are realized in terms of oscillators in our approach.

Let us remark that the invariant given in (2.79) with the coefficients (2.83) can be expressed as a Gauß hypergeometric function ${}_2F_1(a,b;c;x)$,

$$\begin{aligned}|\Psi_{4,2}\rangle = &\frac{1}{c_3!c_4!\Gamma(1-c_3+z)}\\ &\cdot {}_2F_1\left(-c_3,-c_4,1-c_3+z,\frac{(2\bullet 3)(1\bullet 4)}{(1\bullet 3)(2\bullet 4)}\right)(1\bullet 3)^{c_3}(2\bullet 4)^{c_4}|0\rangle\,.\end{aligned} \tag{2.88}$$

Besides the invariant (2.79) that we identified with an R-matrix, there are further Yangian invariants that can be obtained from the four-site monodromy (2.76). For these solutions the conditions on the representation labels in (2.77) are relaxed to $c_1+c_2+c_3+c_4=0$. However, they to not depend on a complex spectral parameter.

2.4.2 Non-Compact Supersymmetric Invariants

The aim of this section is to generalize the compact bosonic sample invariants to the non-compact supersymmetric case. Hence we use the oscillator representations of $\mathfrak{u}(p,q|m=r+s)$ of section 2.3 in full generality. Again we work with a monodromy that has K dual sites left of $N-K$ ordinary ones,

$$\begin{aligned}M_{N,K}(u) =&R_{\square\,\bar{\mathcal{D}}_{c_1}}(u-v_1)\cdots R_{\square\,\bar{\mathcal{D}}_{c_K}}(u-v_K)\\ &\cdot R_{\square\,\mathcal{D}_{c_{K+1}}}(u-v_{K+1})\cdots R_{\square\,\mathcal{D}_{c_N}}(u-v_N)\,.\end{aligned} \tag{2.89}$$

The Lax operators are given in (2.18). They contain the $\mathfrak{gl}(n|m)$ generators $J_{\mathcal{AB}}=\bar{\mathbf{J}}_{\mathcal{AB}}$ from (2.36) at the sites with a dual representation of the type $\bar{\mathcal{D}}_c$ and $J_{\mathcal{AB}}=\mathbf{J}_{\mathcal{AB}}$ from (2.34) at sites with an ordinary representation $\mathcal{D}_c$. Recall that in the non-compact case the representation label c can be any integer. In all the examples discussed in the previous section the overall normalization of the monodromy reduced to unity. Therefore we choose the normalization $f_{\mathcal{D}_c}=f_{\bar{\mathcal{D}}_c}=1$ directly at the level of the Lax operators. The solutions $|\Psi_{N,K}\rangle$ to the Yangian invariance condition (2.24), that we will present momentarily, will be expressed in terms of the contractions of oscillators

$$\begin{aligned}(k\bullet l) &= \sum_{\mathsf{A}}\bar{\mathbf{A}}^l_{\mathsf{A}}\bar{\mathbf{A}}^k_{\mathsf{A}} = \sum_{\alpha=1}^{p}\bar{\mathbf{a}}^l_\alpha\bar{\mathbf{a}}^k_\alpha+\sum_{a=1}^{r}\bar{\mathbf{c}}^l_a\bar{\mathbf{c}}^k_a\,,\\ (k\circ l) &= \sum_{\dot{\mathsf{A}}}\bar{\mathbf{A}}^l_{\dot{\mathsf{A}}}\bar{\mathbf{A}}^k_{\dot{\mathsf{A}}} = \sum_{\dot\alpha=1}^{q}\bar{\mathbf{b}}^l_{\dot\alpha}\bar{\mathbf{b}}^k_{\dot\alpha}+\sum_{\dot a=1}^{s}\bar{\mathbf{d}}^l_{\dot a}\bar{\mathbf{d}}^k_{\dot a}\,.\end{aligned} \tag{2.90}$$

Here site l carries an ordinary representation $\mathcal{D}_{c_l}$ and site k a dual representation $\bar{\mathcal{D}}_{c_k}$. Notice that the contractions are bosonic because they contain fermionic oscillators only in quadratic terms. Furthermore, they are $\mathfrak{gl}(p|r)$ and $\mathfrak{gl}(q|s)$ invariant, respectively,

$$[\bar{\mathbf{J}}^k_{\mathsf{AB}}+\mathbf{J}^l_{\mathsf{AB}},(k\bullet l)\}=0\,,\quad [\bar{\mathbf{J}}^k_{\dot{\mathsf{A}}\dot{\mathsf{B}}}+\mathbf{J}^l_{\dot{\mathsf{A}}\dot{\mathsf{B}}},(k\circ l)\}=0\,. \tag{2.91}$$

They generalize the oscillator contraction given in (2.61) that we used to build the compact bosonic invariants.

2.4.2.1 Two-Site Invariant

We consider the monodromy $M_{2,1}(u)$. To proceed we make an ansatz for the Yangian invariant state $|\Psi_{2,1}\rangle$ as a power series in $(1 \bullet 2)$ and $(1 \circ 2)$ acting on the Fock vacuum $|0\rangle$. Next, we demand Yangian invariance (2.24) of this ansatz. Furthermore, we impose that each site carries an irreducible representation of $\mathfrak{u}(p,q|r+s)$, i.e. $\bar{\mathbf{C}}^1|\Psi_{2,1}\rangle = c_1|\Psi_{2,1}\rangle$ and $\mathbf{C}^2|\Psi_{2,1}\rangle = c_2|\Psi_{2,1}\rangle$. This fixes the invariant, up to a normalization constant, to be

$$
\begin{aligned}
|\Psi_{2,1}\rangle &= \sum_{\substack{g_{12},h_{12}=0\\ g_{12}-h_{12}=q-s-c_1}}^{\infty} \frac{(1 \bullet 2)^{g_{12}}}{g_{12}!}\frac{(1 \circ 2)^{h_{12}}}{h_{12}!}|0\rangle \\
&= \frac{I_{q-s-c_1}\left(2\sqrt{(1 \bullet 2)(1 \circ 2)}\right)}{\sqrt{(1 \bullet 2)(1 \circ 2)}^{\,q-s-c_1}}(1 \bullet 2)^{q-s-c_1}|0\rangle \,,
\end{aligned}
\tag{2.92}
$$

where we identified the sum with the series expansion of the modified Bessel function of the first kind $I_\nu(x)$.[6] The parameters of the monodromy have to obey

$$
v_1 - v_2 = 1 - n + m - c_2\,, \quad c_1 + c_2 = 0\,. \tag{2.93}
$$

We observe that the invariant (2.92) can be expressed as the complex contour integral

$$
|\Psi_{2,1}\rangle = \frac{1}{2\pi i}\oint \mathrm{d}C_{12}\frac{e^{C_{12}(1\bullet 2)+C_{12}^{-1}(1\circ 2)}|0\rangle}{C_{12}^{1+q-s-c_1}}\,. \tag{2.94}
$$

Here the contour is a counterclockwise unit circle around the essential singularity at $C_{12} = 0$. It can be interpreted as group manifold of the unitary group $U(1)$. The integral is easily evaluated using the residue theorem. This yields the series representation in (2.92). As we will see in section 4.1, (2.94) can be considered as a simple Graßmannian integral. In that section we will generalize this simple formula to a Graßmannian integral formulation of the Yangian invariants $|\Psi_{N=2K,K}\rangle$ with oscillator representations.

We finish this section with some remarks. We note that recently a two-site Yangian invariant for oscillator representations of $\mathfrak{psu}(2,2|4)$ was used in [147] based on a construction in [148]. It takes the form of an exponential function instead of a Bessel function as in (2.92). This difference occurs because the sites of that invariant of [147] do not transform in irreducible representations of the symmetry algebra, i.e. in our terminology this invariant would not be an eigenstate of $\bar{\mathbf{C}}^1$ and $\mathbf{C}^2$.[7] Furthermore, we remark that employing the identity

$$
\frac{I_\nu(2\sqrt{x})}{\sqrt{x}^{\,\nu}} = \frac{{}_0F_1(\nu+1;x)}{\Gamma(\nu+1)}\,, \tag{2.95}
$$

cf. [149], the invariant (2.92) can alternatively be expressed in terms of a generalized hypergeometric function ${}_0F_1(a;x)$. Sometimes this form is more convenient because it avoids the "spurious" square roots, which are absent in the series expansion. Additionally, the invariant in (2.92) has infinite norm and thus is technically speaking not an element of $\bar{\mathcal{D}}_{c_1} \otimes \mathcal{D}_{c_2}$ considered as a Hilbert space. As a last aside, let us consider the special case of the compact bosonic algebra $\mathfrak{u}(n=p,0|0)$, i.e. we set $q = r = s = 0$. Then the sum in (2.92) simplifies to a single term

$$
|\Psi_{2,1}\rangle = \frac{(1 \bullet 2)^{c_2}}{c_2!}|0\rangle \tag{2.96}
$$

[6] In the double sum in (2.92) the expression $q - s - c_1$ can also take negative values. The validity of the Bessel function formulation in this case is easily verified using the series expansion.

[7] We thank Ivan Kostov and Didina Serban for clarifying this point.

with $c_2 \geq 0$, where we used $(1\circ 2)^h = \delta_{0h}$. Up to a normalization factor, this is the compact bosonic two-site Yangian invariant known from (2.61).

2.4.2.2 Three-Site Invariants

The generalization of the compact bosonic three-site invariants $|\Psi_{3,1}\rangle$ and $|\Psi_{3,2}\rangle$ to the non-compact supersymmetric setting is not as straightforward as for two sites. Therefore we start by focusing on a particular case. For the non-compact bosonic algebra $\mathfrak{u}(p,1)$ we verify by an explicit computation that the vector

$$|\Psi_{3,1}\rangle = \bar{\mathbf{A}}_{p+1}(1\circ 2)^{-(c_2+1)}(1\circ 3)^{-(c_3+1)}\sum_{k=0}^{\infty}\frac{\big((1\bullet 2)(1\circ 2)+(1\bullet 3)(1\circ 3)\big)^k}{k!\big(-(c_2+1)-(c_3+1)+1+k\big)!}|0\rangle \quad (2.97)$$

solves the Yangian invariance condition (2.24) with the monodromy $M_{3,1}(u)$ in (2.89). For this the parameters of the monodromy have to obey

$$\begin{gathered} v_1 - v_3 = 1 - n - c_3\,, \quad v_1 - v_2 = 1 - n - c_2 - c_3\,, \\ c_1 + c_2 + c_3 = 0\,, \quad c_2 + 1 \leq 0\,, \quad c_3 + 1 \leq 0\,. \end{gathered} \quad (2.98)$$

Notice that this invariant cannot be expressed entirely in terms of $(k\bullet l)$ and $(k\circ l)$ but contains the individual oscillator $\bar{\mathbf{A}}_{p+1}$. Furthermore, note that one can easily identify (2.97) with the series expansion of the hypergeometric function ${}_0F_1(a;x)$.

The invariant in (2.97) immediately raises the question about the generalization to other algebras. The $\mathfrak{u}(2,2)$ case is of special interest because of its relevance for amplitudes. We did not manage to find solutions $|\Psi_{3,1}\rangle$ of the Yangian invariance condition for this algebra. Bearing in mind the discussion of three-particle scattering amplitudes around (1.29), this was to be expected. We argued that these amplitudes do not exist for real particle momenta. Although (1.29) are superamplitudes, the same holds true for the purely bosonic three-particle amplitudes. The $\mathfrak{u}(2,2)$ representations in terms of spinor helicity variables are equivalent to those in terms of oscillators employed here, see [150] and also [151] as well as section 4.2.2 below. This means there should be no $|\Psi_{3,1}\rangle$. Let us briefly sketch an alternative argument supporting this conclusion. It is based on the decomposition of the tensor product of two oscillator representations $\mathcal{D}_{c_2}\otimes\mathcal{D}_{c_3}$ into a sum of irreducible representations of $\mathfrak{u}(2,2)$. For the invariant $|\Psi_{3,1}\rangle\in\bar{\mathcal{D}}_{c_1}\otimes\mathcal{D}_{c_2}\otimes\mathcal{D}_{c_3}$ to exist, this sum must contain an oscillator representation, namely $\mathcal{D}_{-c_1}$ that is dual to $\bar{\mathcal{D}}_{c_1}$. However, from (4.12) and (4.13) of [152] one concludes that the tensor product decomposition of two $\mathfrak{u}(2,2)$ representations that are each built from one family of oscillators does not contain the same type of oscillator representation. This explains why we were not able to solve the Yangian invariance condition.

Let us remark that the formulas of [152] that we used for our argument are just a special case of the decomposition of the multi-fold tensor product for $\mathfrak{u}(p,q)$ oscillator representations in [141]. With this reference it should be straightforward to analyze for which bosonic algebras and representations labels the three-site invariants $|\Psi_{3,1}\rangle$ and $|\Psi_{3,2}\rangle$ as well as further invariants $|\Psi_{N,K}\rangle$ can exist. The tensor product decomposition for superalgebras including $\mathfrak{u}(p,q|m)$ was worked out in [153].

2.4.2.3 Four-Site Invariant

In contrast to the situation for three sites, in case of the four-site invariant $|\Psi_{4,2}\rangle$ the generalization to the non-compact superalgebra $\mathfrak{u}(p,q|m)$ is possible again. It is given by the unwieldy formula

$$
\begin{aligned}
|\Psi_{4,2}\rangle = \sum_{\substack{g_{13},\ldots,g_{24}=0\\ h_{13},\ldots,h_{24}=0\\ \text{with (2.100)}}}^{\infty} & \frac{(1\bullet 3)^{g_{13}}}{g_{13}!}\frac{(1\bullet 4)^{g_{14}}}{g_{14}!}\frac{(2\bullet 3)^{g_{23}}}{g_{23}!}\frac{(2\bullet 4)^{g_{24}}}{g_{24}!}\\
\cdot & \frac{(1\circ 3)^{h_{13}}}{h_{13}!}\frac{(1\circ 4)^{h_{14}}}{h_{14}!}\frac{(2\circ 3)^{h_{23}}}{h_{23}!}\frac{(2\circ 4)^{h_{24}}}{h_{24}!}|0\rangle\\
\cdot & (-1)^{g_{14}+h_{14}}\mathrm{B}(g_{14}+h_{23}+1, h_{13}+g_{24}-v_1+v_2)\,.
\end{aligned}
\tag{2.99}
$$

In this expression the summation range is constrained by

$$
\begin{aligned}
g_{13}-h_{13}+g_{14}-h_{14} &= -c_1+q-s\,, & g_{23}-h_{23}+g_{24}-h_{24} &= -c_2+q-s\,,\\
g_{13}-h_{13}+g_{23}-h_{23} &= c_3+q-s\,, & g_{14}-h_{14}+g_{24}-h_{24} &= c_4+q-s\,.
\end{aligned}
\tag{2.100}
$$

These constraints assure that the eigenvalues of $\bar{\mathbf{C}}^1, \bar{\mathbf{C}}^2, \mathbf{C}^3, \mathbf{C}^4$ acting on the invariant are, respectively, c_1, c_2, c_3, c_4. Furthermore, we made use of the Euler beta function

$$
\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,.
\tag{2.101}
$$

To ensure the Yangian invariance of (2.99), the parameters of the monodromy (2.89) have to obey

$$
v_1-v_3 = 1-n+m-c_3\,,\quad v_2-v_4 = 1-n+m-c_4\,,\quad c_1+c_3=0\,,\quad c_2+c_4=0\,.
\tag{2.102}
$$

For all compact and non-compact sample invariants discussed until here the Yangian invariance condition (2.24) can be verified by a straightforward explicit calculation. Of course, this is also possible for the invariant $|\Psi_{4,2}\rangle$ in (2.99). However, the complexity of the formula already foreshadows that such a computation is somewhat laborious in this case. Therefore we do not display it here. Instead we refer the reader to section 4.1.5 where (2.99) is derived from a Yangian invariant Graßmannian integral.

It is worth noting that in the compact bosonic case $\mathfrak{u}(n=p,0|0)$ the invariant (2.99) simplifies to

$$
\begin{aligned}
|\Psi_{4,2}\rangle = \sum_{g_{14}=0}^{\infty} & \frac{(1\bullet 3)^{c_3-g_{14}}}{(c_3-g_{14})!}\frac{(1\bullet 4)^{g_{14}}}{g_{14}!}\frac{(2\bullet 3)^{g_{14}}}{g_{14}!}\frac{(2\bullet 4)^{c_4-g_{14}}}{(c_4-g_{14})!}|0\rangle\\
\cdot & (-1)^{g_{14}}\mathrm{B}(g_{14}+1, -v_3+v_4+c_3-g_{14})\,.
\end{aligned}
\tag{2.103}
$$

This agrees with the compact invariant in (2.79) up to a normalization factor. An interesting question is whether the formulation of this compact invariant as a hypergeometric function in (2.88) generalizes to the non-compact invariant (2.99).

Finally, let us mention that one can also work out an intertwiner version $\mathcal{O}_{\Psi_{4,2}}$ of the non-compact supersymmetric invariant $|\Psi_{4,2}\rangle$ in (2.99). As shown in detail for the compact bosonic case in (2.87), this intertwiner corresponds to an R-matrix satisfying a Yang-Baxter equation. This R-matrix was worked out explicitly in [104] employing the same oscillator representations of $\mathfrak{u}(p,q|m)$ that we use here. For the algebra $\mathfrak{u}(2,2|4)$ it is essentially the R-matrix of the spin chain governing the planar $\mathcal{N}=4$ SYM one-loop spectral problem [39, 40]. A word of caution is in order here. The oscillator representations used in this thesis and in [104] are those of [140], see also [154] for the $\mathfrak{u}(2,2|4)$ case. The bosonic generators $\bar{\mathbf{a}}_\alpha\mathbf{a}_\beta$ and $\bar{\mathbf{b}}_{\dot\alpha}\mathbf{b}_{\dot\beta}$, cf. (2.32), are associated with the compact subalgebra $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ of the conformal algebra $\mathfrak{su}(2,2)$. In contrast, the generators named $\bar{\mathbf{a}}_\alpha\mathbf{a}_\beta$ and $\bar{\mathbf{b}}_{\dot\alpha}\mathbf{b}_{\dot\beta}$ in [40] are associated with the non-compact Lorentz subalgebra $\mathfrak{sl}(2)\equiv\mathfrak{sl}(\mathbb{C}^2)$ of

$\mathfrak{su}(2,2)$. Thus the "oscillators" in [40] satisfy non-standard reality conditions, see e.g. [155]. Some clarifications on related issues can be found in [156]. The difference between the oscillator representation used in this thesis and in [104] versus that in [40] does not seem to affect the R-matrix. Probably this is because of its $\mathfrak{gl}(4|4)$ invariance. Leaving aside this subtlety, the connection between Yangian invariance and the integrability discovered in the $\mathcal{N}=4$ SYM spectral problem is a good point to finish this chapter.

Chapter 3

Bethe Ansätze and Vertex Models

In the present chapter we employ the technology introduced in chapter 2 for a systematic construction of Yangian invariants with finite-dimensional $\mathfrak{gl}(n)$ representations. In particular, the QISM formulation of the Yangian algebra proves to be well suited for the construction of such invariants by means of Bethe ansatz methods. Furthermore, we explain that the partition functions of a rather general class of vertex models can be interpreted as Yangian invariants.

Let us outline the content of this chapter in more detail. The Bethe ansatz is a powerful technique to solve integrable spin chain models. The arguably simplest model to which it applies is the Heisenberg spin chain, that is based on the Yangian of $\mathfrak{gl}(2)$, cf. section 1.1. In section 3.1 we review its solution using an algebraic Bethe ansatz, which makes crucial use of the QISM. The main results of this chapter are contained in section 3.2. We argue that Yangian invariants for $\mathfrak{gl}(n)$ are particular eigenstates of special spin chains and therefore in principle accessible by a Bethe ansatz. We detail our argument in the $\mathfrak{gl}(2)$ case by using the algebraic Bethe ansatz to construct Yangian invariants. This leads to a characterization of those Yangian invariants in terms of certain functional relations. In addition, this Bethe ansatz reproduces the compact bosonic sample invariants of section 2.4.1. We also explain a classification of the solutions to the functional relations. This classification is of relevance more generally for non-compact supersymmetric Yangian invariants and tree-level amplitudes of planar $\mathcal{N} = 4$ SYM. Further results on the Bethe ansatz for Yangian invariants, in particular concerning its extension to the $\mathfrak{gl}(n)$ case, have been shifted to appendix A.

A different avenue is explored in the remainder of the chapter. In section 3.3 we review the rational six-vertex model. The algebraic structure of this two-dimensional classical statistical model is closely related to that of the quantum Heisenberg spin chain. To be precise, both models are based on the same $\mathfrak{gl}(2)$ symmetric solution of the Yang-Baxter equation. The partition function of this vertex model, even on a non-rectangular lattice, can be computed by the perhaps little known perimeter Bethe ansatz. In section 3.4 we show that the partition functions of even more general vertex models with $\mathfrak{gl}(n)$ symmetry correspond to a certain class of Yangian invariants. Combined with the results of the previous sections, this allows us to understand our Bethe ansatz for Yangian invariants as a generalization of the perimeter Bethe ansatz.

3.1 Bethe Ansatz for Heisenberg Spin Chain

In 1928 Heisenberg introduced a mathematical model of ferromagnetism [157]. It consists of electron spins on a lattice that interact with their nearest neighbors. In the one-dimensional case it degenerates into a linear chain of spins, the so-called *Heisenberg spin chain*. Its Hamiltonian

$$\mathscr{H} = -\sum_{i=1}^{N} \vec{\sigma}^{\,i} \cdot \vec{\sigma}^{\,i+1} \tag{3.1}$$

acts on the Hilbert space $(\mathbb{C}^2)^{\otimes N}$ and we assume periodicity, $\vec{\sigma}^{\,i+N} = \vec{\sigma}^{\,i}$. The spins are modeled by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.2}$$

that generate the Lie algebra $\mathfrak{su}(2)$. Remarkably, this system can be *solved exactly*, which here means that there exist efficient analytical methods to diagonalize the Hamiltonian. This was achieved by Bethe in 1931 using a technique nowadays called *coordinate Bethe ansatz* [158]. Let us emphasize that although the seemingly simple Heisenberg spin chain has been under investigation for the better part of a century, it is still an active field of research. Some more recent developments are reviewed in [159]. Theoretically, the exact solvability extends beyond the energy spectrum, shedding light also on the structure of correlation functions. The model can even be probed experimentally as one-dimensional magnetic chains are realized in certain crystals.

In this section we discuss the diagonalization of the Heisenberg Hamiltonian (3.1) by means of an *algebraic Bethe ansatz*, see the authoritative reviews [14, 15]. It provides an alternative to Bethe's original technique. We choose this flavor of the Bethe ansatz because it is deeply rooted in the QISM, that we used to discuss the Yangian algebra in section 2.1. Furthermore, it straightforwardly generalizes to a large class of more elaborate spin chain models.

In order to apply the algebraic Bethe ansatz, we have to rephrase the Hamiltonian (3.1) in the QISM language. This may seem like a detour at first but it will pay off eventually. We start with a monodromy matrix (2.19) associated with the Yangian of $\mathfrak{gl}(2)$,

$$M(u) = R_{\square\,\square_1}(u) \cdots R_{\square\,\square_N}(u)\,. \tag{3.3}$$

Here we specialized to the case of a homogeneous spin chain with $v_i = 0$. In addition, we chose the defining representation of $\mathfrak{gl}(2)$ for each local quantum space $\mathcal{V}_i = \square_i = \mathbb{C}^2$. The generators $J^i_{\alpha\beta} = E^i_{\alpha\beta}$ are specified in (2.2). In accordance with section 2.3 we use Greek letters for the bosonic indices. The total quantum space $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_N$ already matches that of the Heisenberg chain. Next, we introduce a *transfer matrix* by taking the trace over the auxiliary space $\square$,

$$T(u) = \operatorname{tr}_\square M(u)\,. \tag{3.4}$$

This operator acts solely on the total quantum space and it comprises the Hamiltonian (3.1). To show this, we fix the normalization of the Lax operators (2.18) entering the monodromy matrix to be $f_{\square_i}(u) = u$.[1] At a special value of the spectral parameter these operators reduce to permutation operators on $\square \otimes \square_i$,

$$R_{\square\,\square_i}(u)\Big|_{u=0} = \sum_{\alpha,\beta=1}^{2} E_{\alpha\beta} E^i_{\beta\alpha} =: P_{\square\,\square_i}\,. \tag{3.5}$$

[1] This choice does not respect the condition (2.14) but is most convenient for this section.

Such permutation operators satisfy

$$P_{\square_i \square_j} P_{\square_i \square_k} = P_{\square_j \square_k} P_{\square_i \square_j}\,, \quad P_{\square_i \square_j} = P_{\square_j \square_i}\,, \quad P_{\square_i \square_j} = P^{-1}_{\square_i \square_j}\,. \tag{3.6}$$

These properties allow us to extract a shift operator from the transfer matrix,

$$T(u)\Big|_{u=0} = P_{\square_N \square_{N-1}} \cdots P_{\square_3 \square_2} P_{\square_2 \square_1} =: e^{\mathscr{P}}\,, \tag{3.7}$$

which we express in terms of its generator $\mathscr{P}$. The Heisenberg Hamiltonian (3.1) is obtained by taking the logarithmic derivative,

$$T(u)^{-1} \frac{\mathrm{d}}{\mathrm{d}u} T(u)\Big|_{u=0} = \sum_{i=1}^{N} P_{\square_i \square_{i+1}} = \frac{1}{2}(-\mathscr{H} + N) \tag{3.8}$$

with the identification $\square_{i+N} = \square_i$. Thus in the expansion of the transfer matrix with respect to the spectral parameter u,

$$T(u) = \exp\left(\mathscr{Q}^{[0]} + u\mathscr{Q}^{[1]} + u^2\mathscr{Q}^{[2]} + \dots\right), \tag{3.9}$$

we can identify the first two coefficients with the generator of the shift and the Hamiltonian, respectively,

$$\mathscr{Q}^{[0]} = \mathscr{P}\,, \quad \mathscr{Q}^{[1]} = \frac{1}{2}(-\mathscr{H} + N)\,. \tag{3.10}$$

The algebraic Bethe ansatz is a method to diagonalize the transfer matrix. This matrix commutes for different values of the spectral parameter,

$$[T(u), T(u')] = 0\,. \tag{3.11}$$

This follows from the Yang-Baxter-like defining relation (2.9) of the Yangian after taking the trace over the auxiliary spaces $\square$ and $\square'$. It implies that all the coefficients of the expansion in (3.8) commute,

$$[\mathscr{Q}^{[r]}, \mathscr{Q}^{[s]}] = 0\,. \tag{3.12}$$

Therefore the algebraic Bethe ansatz in particular diagonalizes the Hamiltonian (3.1) of the Heisenberg spin chain. Let us remark that the expansion coefficients $\mathscr{Q}^{[r]}$ are denoted as "conserved quantities" because the Hamiltonian is among them. Furthermore, we may say that all these charges are "in involution" because they commute. A sufficient number of conserved quantities in involution, i.e. that Poisson-commute, is the key ingredient of the Liouville theorem, that defines integrable models with finitely many degrees of freedom in classical mechanics, cf. [6, 7]. In this sense, the QISM can be understood as a quantization of that classical theorem, as already pointed out in section 1.1.

After embedding the one-dimensional Heisenberg model in the QISM, we discuss its solution via the algebraic Bethe ansatz. In fact, we present this method for a more general class of $\mathfrak{gl}(2)$ spin chains. Our presentation highlights the essential features. More detailed expositions can be found in the aforementioned reviews [14, 15]. We employ a $\mathfrak{gl}(2)$ monodromy matrix (2.19) with general finite-dimensional representations $\mathcal{V}_i$, inhomogeneity parameters v_i and normalizations $f_{\mathcal{V}_i}$ at the sites. The monodromy (3.3) of the Heisenberg model is a special case thereof. We express the general $\mathfrak{gl}(2)$ monodromy as a matrix in the auxiliary space $\square = \mathbb{C}^2$,

$$M(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{3.13}$$

The entries of this matrix are operators on the total quantum space $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_N$ of the spin chain. This notation leads to the transfer matrix

$$T(u) = \mathrm{tr}_\square\, M(u) = A(u) + D(u)\,. \tag{3.14}$$

In what follows, we use the Yang-Baxter equation (2.9) obeyed by the monodromy $M(u)$ to diagonalize this transfer matrix. We assume the existence of a reference state $|\Omega\rangle$ that is characterized by

$$A(u)|\Omega\rangle = \alpha(u)|\Omega\rangle\,, \quad D(u)|\Omega\rangle = \delta(u)|\Omega\rangle\,, \quad C(u)|\Omega\rangle = 0 \tag{3.15}$$

with some scalar functions $\alpha(u)$ and $\delta(u)$. These conditions are fulfilled if we choose finite-dimensional $\mathfrak{gl}(2)$ representations $\mathcal{V}_i$ for the generators $J^i_{\alpha\beta}$ in the Lax operators (2.18) at the spin chain sites. Such representations contain a highest weight state $|\sigma_i\rangle$ that obeys

$$J^i_{11}|\sigma_i\rangle = \xi_i^{(1)}|\sigma_i\rangle\,, \quad J^i_{22}|\sigma_i\rangle = \xi_i^{(2)}|\sigma_i\rangle\,, \quad J^i_{12}|\sigma_i\rangle = 0\,. \tag{3.16}$$

The scalar coefficients in these equations may be arranged into a highest weight vector $\Xi_i = (\xi_i^{(1)}, \xi_i^{(2)})$ that characterizes the representation. In case of the Heisenberg model with the defining representation $\mathcal{V}_i = \square_i$, we have $\Xi_i = (1, 0)$ and $|\sigma_i\rangle = \binom{0}{1}$. For a monodromy built from representations characterized by (3.16) the reference state in (3.15) becomes the tensor product of the highest weight states at the individual sites,

$$|\Omega\rangle = |\sigma_1\rangle \otimes \cdots \otimes |\sigma_N\rangle\,. \tag{3.17}$$

Moreover, the scalar functions in (3.15) can be computed,

$$\alpha(u) = \prod_{i=1}^{N} f_{\mathcal{V}_i}(u - v_i)\frac{u - v_i + \xi_i^{(1)}}{u - v_i}\,, \quad \delta(u) = \prod_{i=1}^{N} f_{\mathcal{V}_i}(u - v_i)\frac{u - v_i + \xi_i^{(2)}}{u - v_i}\,. \tag{3.18}$$

However, in the following we do not use this explicit form of the functions. It is sufficient to demand (3.15). We continue by making an ansatz for the eigenstates of the transfer matrix (3.14),

$$|\Psi\rangle = B(u_1)\cdots B(u_P)|\Omega\rangle\,. \tag{3.19}$$

It depends on P complex parameters u_k, the so-called *Bethe roots*. For generic values of these parameters the ansatz is not an eigenstate of the transfer matrix. However, it turns into one if the parameters satisfy a set of algebraic equations referred to as *Bethe equations*. To derive these equations we need some commutation relations among the elements of the monodromy (3.13), which follow from the defining relation (2.9),

$$\begin{aligned}
A(u)B(u') &= \frac{u - u' - 1}{u - u'}B(u')A(u) + \frac{1}{u - u'}B(u)A(u')\,,\\
D(u)B(u') &= \frac{u - u' + 1}{u - u'}B(u')D(u) - \frac{1}{u - u'}B(u)D(u')\,,\\
B(u)B(u') &= B(u')B(u)\,.
\end{aligned} \tag{3.20}$$

We continue by acting with the operators $A(u)$ and $D(u)$ on the ansatz (3.19). Using (3.20) we commute these operator to the right and once they hit the reference state we employ (3.15). After a laborious calculation this yields, see e.g. [14],

$$
\begin{aligned}
A(u)|\Psi\rangle &= \alpha(u)\frac{Q(u-1)}{Q(u)}|\Psi\rangle - \sum_{k=1}^{P}\frac{\alpha(u_k)Q(u_k-1)}{u-u_k}B(u)\prod_{\substack{i=1\\ i\neq k}}^{P}\frac{B(u_i)}{u_k-u_i}|\Omega\rangle\,,\\
D(u)|\Psi\rangle &= \delta(u)\frac{Q(u+1)}{Q(u)}|\Psi\rangle - \sum_{k=1}^{P}\frac{\delta(u_k)Q(u_k+1)}{u-u_k}B(u)\prod_{\substack{i=1\\ i\neq k}}^{P}\frac{B(u_i)}{u_k-u_i}|\Omega\rangle\,.
\end{aligned}
\tag{3.21}
$$

We expressed the result using Baxter's Q-function, that is a polynomial of degree P whose roots are the Bethe roots u_k,

$$
Q(u) = \prod_{k=1}^{P}(u-u_k)\,. \tag{3.22}
$$

To render $|\Psi\rangle$ into an eigenstate of the transfer matrix (3.14), the Bethe equations

$$
\alpha(u_k)Q(u_k-1) + \delta(u_k)Q(u_k+1) = 0 \tag{3.23}
$$

for $k = 1, \ldots, P$ have to be obeyed. They guarantee that after summing up the two lines in (3.21) the "unwanted terms", which are not proportional to $|\Psi\rangle$, disappear. A more common form of the Bethe equations is obtained by solving for the fraction of Q-functions and using (3.18) as well as (3.22),

$$
\prod_{i=1}^{N}\frac{u_k - v_i + \xi_i^{(1)}}{u_k - v_i + \xi_i^{(2)}} = -\prod_{j=1}^{P}\frac{u_k-u_j+1}{u_k-u_j-1}\,. \tag{3.24}
$$

However, we will mostly work with the Bethe equations in the form (3.23) because for the special solutions that we will examine in section 3.2, we would divide by zero in (3.24). From (3.21) it follows that the eigenvalue $\tau(u)$ of $T(u)$ for the eigenstate $|\Psi\rangle$ is given by the *Baxter equation*

$$
\tau(u) = \alpha(u)\frac{Q(u-1)}{Q(u)} + \delta(u)\frac{Q(u+1)}{Q(u)}\,. \tag{3.25}
$$

The Bethe equations are a consequence of this equation alone if one assumes regularity of the functions $\tau(u)$, $\alpha(u)$ and $\delta(u)$ at the values of Bethe roots $u = u_k$ and furthermore demands $Q(u)$ to be of the form (3.22). With these assumptions taking the residues of (3.25) at $u = u_k$ yields the Bethe equations (3.23).

In this section we demonstrated how the transfer matrix $T(u)$ in (3.14) of a finite-dimensional $\mathfrak{gl}(2)$ spin chain, and in particular of the Heisenberg model, is diagonalized employing the algebraic Bethe ansatz. This method led to the formula (3.19) for the eigenvectors $|\Psi\rangle$ of the transfer matrix and to (3.25) for its eigenvalues $\tau(u)$. Both formulas are parameterized in terms of Bethe roots u_k, which have to obey the Bethe equations (3.23). Obtaining solutions of these algebraic equations is an important and difficult part of the analysis of the integrable model at hand. For this one often has to resort to numerical approximations. However, in certain cases exact analytical solutions can be obtained. In the next section we will study in detail a situation where this is possible. Let us also add that while we focused on $\mathfrak{gl}(2)$ spin chains, the algebraic Bethe ansatz can be extended to $\mathfrak{gl}(n)$. This higher rank case is referred to as *nested* algebraic Bethe ansatz and it is technically considerably more involved, see e.g. [160].

3.2 Bethe Ansatz for Yangian Invariants

We concluded chapter 2 by presenting some sample Yangian invariants. These were constructed "by hand" in the sense that we explicitly inspected the Yangian invariance condition. The aim of this section is to put forward a systematic construction of such Yangian invariants. To explain the basic idea, let us focus on the bosonic case of the Yangian of $\mathfrak{gl}(n)$. The crucial starting point of this method is the Yangian invariance condition (2.24) in terms of a spin chain monodromy $M(u)$ rather than the expanded versions (2.27) or (2.28) thereof. Taking the trace over the auxiliary space $\square = \mathbb{C}^n$ in this formulation of the Yangian invariance condition yields

$$T(u)|\Psi\rangle = n|\Psi\rangle \tag{3.26}$$

with the transfer matrix

$$T(u) = \operatorname{tr}_\square M(u)\,. \tag{3.27}$$

Therefore a Yangian invariant $|\Psi\rangle$ is a special eigenstate of a transfer matrix. This transfer matrix can be diagonalized using a Bethe ansatz, at least for finite-dimensional representations in the quantum space. This renders the invariant $|\Psi\rangle$ into a special Bethe vector and thereby makes it amenable to a Bethe ansatz construction.

Here we will implement this general idea for simplicity in case of compact invariants of the Yangian of $\mathfrak{gl}(2)$. In section 3.2.1 we specialize the Bethe ansatz of section 3.1 to the case of Yangian invariant Bethe vectors. We find that these are characterized by functional relations, which emerge as a special case of the Baxter equation (3.25). These functional relations determine the Bethe roots and also severely constrain the inhomogeneities and representation labels of the monodromy matrix. Remarkably, we find that these relations can easily be solved analytically. In section 3.2.2 we exemplify this observation by discussing sample invariants that include in particular those of section 2.4.1. These sample invariants illustrate the general structure of the solutions. The Bethe roots form strings in the complex plane. The position of these strings is determined by the inhomogeneities and their length by the representation labels. Section 3.2.3 contains a classification of all solutions to the functional equations and therefore basically of all Yangian invariants within the studied class of representations. Each Yangian invariant is associated with a permutation. These are essentially those permutations that we already encountered in the introductory section 1.3.6 on deformed SYM scattering amplitudes. Supplementary material on the Bethe ansatz for Yangian invariants is provided in appendix A. It contains results on the extension to $\mathfrak{gl}(n)$ and on the evaluation of Yangian invariant Bethe vectors.

3.2.1 Derivation of Functional Relations

To begin with, we write out the definition (2.24) of Yangian invariants for $\mathfrak{gl}(2)$ using the notation (3.13) for the monodromy elements,

$$A(u)|\Psi\rangle = |\Psi\rangle\,, \qquad D(u)|\Psi\rangle = |\Psi\rangle\,, \tag{3.28}$$

$$B(u)|\Psi\rangle = 0\,, \qquad C(u)|\Psi\rangle = 0\,. \tag{3.29}$$

Here we grouped the equations for diagonal elements of the monodromy in (3.28) and those for off-diagonal elements in (3.29). For the construction of solutions to these equations we proceed in two steps. First, we solve (3.28) by specializing the algebraic Bethe ansatz reviewed in section 3.1. Second, we show that for finite-dimensional representations (3.28) implies (3.29). These steps lead to a characterization of Yangian invariants in terms of functional relations, which we summarize at the end of this section.

Let us address the equations in (3.28) for the diagonal monodromy elements. In the Bethe ansatz we seek eigenvectors $|\Psi\rangle$ of the transfer matrix $T(u) = A(u) + D(u)$. Here we demand in addition that $|\Psi\rangle$ is a common eigenvector of $A(u)$ and $D(u)$ individually. We proceed as for usual the Bethe ansatz by making the ansatz (3.19) for $|\Psi\rangle$. Next, and still in full analogy to the Bethe ansatz, we derive the action (3.21) of $A(u)$ and $D(u)$ on this ansatz. The following step differs from the usual Bethe ansatz in that we demand the "unwanted terms" in both lines of (3.21) to vanish individually. This leads to a special case of the Bethe equations (3.23),

$$\alpha(u_k)Q(u_k - 1) = 0\,, \quad \delta(u_k)Q(u_k + 1) = 0\,. \tag{3.30}$$

To obtain the correct eigenvalues of $A(u)$ and $D(u)$ in (3.28), we further have to demand in (3.21) that

$$1 = \alpha(u)\frac{Q(u-1)}{Q(u)}\,, \quad 1 = \delta(u)\frac{Q(u+1)}{Q(u)}\,. \tag{3.31}$$

These equations are a degenerate case of the Baxter equation (3.25). Each of the terms on the right hand site of (3.25) is fixed to 1 and thus we obtain the transfer matrix eigenvalue $\tau(u) = 2$ in accordance with (3.26). Assuming regularity of $\alpha(u)$ and $\delta(u)$ at the Bethe roots $u = u_k$, the residues of the functional relations (3.31) at these points yield the special case of the Bethe equations in (3.30). Consequently, we reduced the solution of the diagonal part (3.28) of the Yangian invariance condition to the functional relations (3.31).

We move on to discuss the equations in (3.29) for the off-diagonal monodromy elements. Our argument makes use of (3.28) for the diagonal elements, which we just solved. Expanding these equations in the spectral parameter u as in (2.13) yields

$$M_{11}^{(1)}|\Psi\rangle = 0\,, \quad M_{22}^{(1)}|\Psi\rangle = 0\,. \tag{3.32}$$

We mentioned after (2.16) that $M_{\beta\alpha}^{(1)}$ are generators of a $\mathfrak{gl}(2)$ algebra. Therefore, (3.32) is equivalent to saying that $|\Psi\rangle$ has $\mathfrak{gl}(2)$ weight $(0, 0)$. Next, we expand $C(u)|\Omega\rangle = 0$ in (3.15) to obtain $M_{21}^{(1)}|\Omega\rangle = 0$. Employing this condition, the commutation relations (2.17) and (3.21) results in

$$M_{21}^{(1)}|\Psi\rangle = -\sum_{k=1}^{P}(\alpha(u_k)Q(u_k - 1) + \delta(u_k)Q(u_k + 1))\prod_{\substack{i=1\\i\neq k}}^{P}\frac{B(u_i)}{u_k - u_i}|\Omega\rangle = 0\,. \tag{3.33}$$

Here we used (3.30) for the last equality. In case of finite-dimensional representations, which we are dealing with, (3.32) and (3.33) are sufficient to guarantee that $|\Psi\rangle$ is a $\mathfrak{gl}(2)$ singlet. This in turn implies

$$M_{12}^{(1)}|\Psi\rangle = 0\,. \tag{3.34}$$

Lastly, we use (2.17) once more to show

$$[M_{12}^{(1)}, A(u) - D(u)] = 2B(u)\,, \quad [M_{21}^{(1)}, D(u) - A(u)] = 2C(u)\,. \tag{3.35}$$

Acting with these equations on $|\Psi\rangle$ and employing (3.28), (3.33) as well as (3.34) implies (3.29), the other part of the Yangian invariance condition for the off-diagonal monodromy elements.

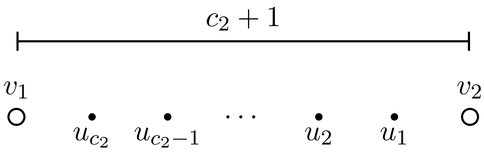


Figure 3.1: The Yangian invariant $|\Psi_{2,1}\rangle$ of section 3.2.2.1 is constructed from Bethe roots u_k that form a string in the complex plane between the inhomogeneities v_1 and v_2, cf. (3.41). The string contains c_2 roots with a uniform real spacing of 1.

To summarize, we reduced the construction of solutions $|\Psi\rangle$ to the Yangian invariance condition (2.24) for $\mathfrak{gl}(2)$ to the functional relations (3.31). Given a solution $(\alpha(u), \delta(u), Q(u))$ of these relations, where $\alpha(u)$ and $\delta(u)$ are regular at $u = u_k$ and $Q(u)$ is of the form (3.22), the Yangian invariant $|\Psi\rangle$ is the Bethe vector (3.19). The Bethe roots in this vector are the zeros of $Q(u)$. There exists are remarkable reformulation of the two functional relations in (3.31). They decouple into one equation that only contains the eigenvalues of the monodromy,

$$1 = \alpha(u)\delta(u-1)\,, \tag{3.36}$$

and another equation in which also the Bethe roots enter,

$$\frac{Q(u)}{Q(u+1)} = \delta(u)\,. \tag{3.37}$$

The crucial step in analyzing this system is to find solutions of (3.36). The eigenvalues of the monodromy depend on the inhomogeneities and the representation labels, cf. (3.18). Consequently, this equation selects those monodromies which admit solutions of the Yangian invariance condition (2.24). Provided a solution of (3.36), the difference equation (3.37) can typically be solved with ease for the Bethe roots u_k contained in $Q(u)$. This is in stark contrast to the usual application of the Bethe ansatz to a spin chain spectral problem, where the Bethe equations are very hard to solve. We refer to the construction of Yangian invariants described in this section as *Bethe ansatz for Yangian invariants*.

3.2.2 Sample Solutions

To get acquainted with the Bethe ansatz construction of Yangian invariants, we use this method to rederive the compact bosonic sample invariants of section 2.4.1 in the $\mathfrak{gl}(2)$ case. The oscillator representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_{-c}$ with $c \in \mathbb{N}$, which we used there at the sites of the monodromy, possess respectively the highest weights $\Xi = (c, 0)$ and $\Xi = (0, -c)$, cf. (2.43). Therefore these invariants fall within the reach of the Bethe ansatz. We discuss the two-site invariant $|\Psi_{2,1}\rangle$, the three-site invariants $|\Psi_{3,1}\rangle$ and $|\Psi_{3,2}\rangle$ as well as the four-site invariant $|\Psi_{4,2}(z)\rangle$ corresponding to the R-matrix. For each of these sample invariants we present the solution of the functional relations (3.36) and (3.37). In particular, we find that the Bethe roots arrange into strings in the complex plane. What is more, we present a superposition principle for solutions of the functional relations.

3.2.2.1 Two-Site Invariant

Here we discuss the $\mathfrak{gl}(2)$ case of the invariant $|\Psi_{2,1}\rangle$ that was constructed "by hand" in section 2.4.1.3. Let us recall the representations and inhomogeneities of the monodromy

$M_{2,1}(u)$ associated with that invariant, cf. (2.58) and (2.59),

$$\begin{gathered}\mathcal{V}_1 = \bar{\mathcal{D}}_{c_1}\,, \quad \mathcal{V}_2 = \mathcal{D}_{c_2}\,,\\ v_1 = v_2 - 1 - c_2\,, \quad c_1 + c_2 = 0\,.\end{gathered} \tag{3.38}$$

We use this data to compute the monodromy eigenvalues in (3.18),

$$\alpha(u) = \frac{u - v_2 + c_2}{u - v_2}\,, \quad \delta(u) = \frac{u - v_2 + 1}{u - v_2 + 1 + c_2}\,, \tag{3.39}$$

where we also inserted the trivial normalization (2.60) and the highest weights (2.43). One readily verifies that these eigenvalues obey the functional relation (3.36). The solution of the remaining relation (3.37) is

$$Q(u) = \frac{\Gamma(u - v_2 + c_2 + 1)}{\Gamma(u - v_2 + 1)} = \prod_{k=1}^{c_2}(u - v_2 + k)\,. \tag{3.40}$$

Demanding the Q-function to be of the polynomial form (3.22) eliminates the freedom to multiply a solution of (3.37) by any function of period 1 in u. Noting that c_2 is a positive integer, the gamma functions reduce to a polynomial. We extract the Bethe roots as its zeros,

$$u_k = v_2 - k \quad \text{for} \quad k = 1, \dots, c_2\,. \tag{3.41}$$

These roots form a string in the complex plane, see figure 3.1. As always for the $\mathfrak{gl}(2)$ Bethe ansatz, we may permute the labels of the Bethe roots because the operators $B(u)$ entering the Bethe vector (3.19) commute for different values of the spectral parameter u, cf. (3.20). Ultimately, we want to obtain the Yangian invariant Bethe vector (3.19) associated with the solution of the functional relations presented here. First of all, this requires the reference state (3.17). For the representations listed in (3.38) it is the tensor product of the highest weight states (2.41),

$$|\Omega\rangle = (\bar{\mathbf{a}}_2^1)^{c_2}(\bar{\mathbf{a}}_1^2)^{c_2}|0\rangle\,. \tag{3.42}$$

Next, we evaluate (3.19) employing (3.38), (3.41) and (3.42). Details of this computation for general $c_2 \in \mathbb{N}$ are presented in appendix A.1. It is worth noting that the normalization of the operators $B(u_k)$ trivializes due to (2.60). We obtain

$$|\Psi\rangle = B(u_1)\cdots B(u_{c_2})|\Omega\rangle = (-1)^{c_2}(1 \bullet 2)^{c_2}|0\rangle \propto |\Psi_{2,1}\rangle\,. \tag{3.43}$$

Thus the Bethe ansatz for Yangian invariants indeed reproduces $|\Psi_{2,1}\rangle$ as computed in (2.61).

3.2.2.2 Three-Site Invariants

Two different tree-site sample invariants were introduced in section 2.4.1.4. The monodromy $M_{3,1}(u)$ associated to the first invariant $|\Psi_{3,1}\rangle$ is defined by the representations and inhomogeneities, cf. (2.64) and (2.65),

$$\begin{gathered}\mathcal{V}_1 = \bar{\mathcal{D}}_{c_1}\,, \quad \mathcal{V}_2 = \mathcal{D}_{c_2}\,, \quad \mathcal{V}_3 = \mathcal{D}_{c_3}\,,\\ v_2 = v_1 + 1 + c_2 + c_3\,, \quad v_3 = v_1 + 1 + c_3\,, \quad c_1 + c_2 + c_3 = 0\,,\end{gathered} \tag{3.44}$$

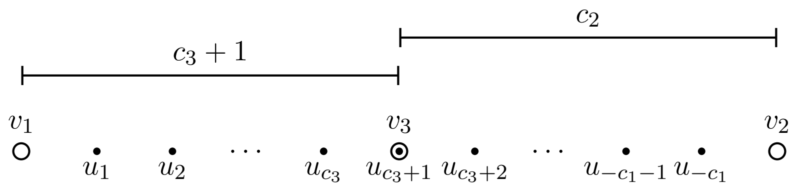


Figure 3.2: The invariant $|\Psi_{3,1}\rangle$ is constructed from a real string of $-c_1 = c_2+c_3 \in \mathbb{N}$ uniformly spaced Bethe roots u_k in the complex plane, cf. (3.47). The roots lie between the inhomogeneities v_1, v_2 and one of them coincides with v_3.

where we specialized to the $\mathfrak{gl}(2)$ case and we recall that $c_2, c_3 \in \mathbb{N}$. With these parameters, the trivial normalization of the monodromy (2.66) and the highest weights (2.43), we evaluate the monodromy eigenvalues (3.18) on the reference state,

$$\alpha(u) = \frac{u - v_1 - 1}{u - v_1 + c_1 - 1}\,, \quad \delta(u) = \frac{u - v_1 + c_1}{u - v_1}\,. \tag{3.45}$$

These satisfy the functional relation (3.36). Assuming the Q-function to be of the form (3.22), the other functional relation (3.37) has the unique solution

$$Q(u) = \frac{\Gamma(u - v_1)}{\Gamma(u - v_1 - c_2 - c_3)} = \prod_{k=1}^{c_2+c_3} (u - v_1 - k)\,. \tag{3.46}$$

Its zeros define the Bethe roots

$$u_k = v_1 + k \quad \text{for} \quad k = 1, \ldots, c_2 + c_3\,, \tag{3.47}$$

which again form a string in the complex plane, see figure 3.2. We continue to calculate the associated Bethe vector. With the representations given in (3.44) the reference state (3.17) for this vector reads

$$|\Omega\rangle = (\bar{\mathbf{a}}_2^1)^{c_2+c_3}(\bar{\mathbf{a}}_1^2)^{c_2}(\bar{\mathbf{a}}_1^3)^{c_3}|0\rangle\,. \tag{3.48}$$

Note that the Bethe root $u_{c_3+1} = v_3$ is identical with an inhomogeneity. As a consequence the Lax operator $R_{\square\mathcal{D}_{c_3}}(u_{c_3+1} - v_3)$, which enters $B(u_{c_3+1})$ in the Bethe vector (3.19), diverges, cf. (2.45). To nevertheless obtain a finite Bethe vector we rely on an ad hoc prescription, which we checked for small values of c_2 and c_3: In a first step, the parameter u_{c_3+1} is kept at a generic value, while all other Bethe roots are inserted into (3.19). This renders the resulting expression finite at $u_{c_3+1} = v_3$. After then also inserting this last root, we are left with

$$|\Psi\rangle = B(u_1)\cdots B(u_{c_2+c_3})|\Omega\rangle = (-1)^{c_2+c_3}(1 \bullet 2)^{c_2}(1 \bullet 3)^{c_3}|0\rangle \propto |\Psi_{3,1}\rangle\,. \tag{3.49}$$

Hence, we constructed the three-site Yangian invariant $|\Psi_{3,1}\rangle$ from (2.67) by means of a Bethe ansatz. It would clearly be desirable to obtain a better understanding of the divergence and to derive (3.49) for general $c_2, c_3 \in \mathbb{N}$. A promising avenue to address these points might be a generalization of appendix A.1 to the three-site case.

So-called "singular solutions" of the Bethe equations, which superficially lead to divergent Bethe vectors, are well-known for the Heisenberg spin chain and certain generalizations thereof. Recent discussions of this phenomenon can be found in [161] and [162], see also the references therein. The problem was even known to Bethe [158]. It also surfaced in the early

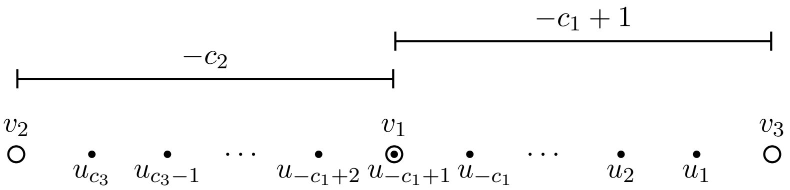


Figure 3.3: The $c_3 = -c_1 - c_2 \in \mathbb{N}$ Bethe roots u_k associated with the invariant $|\Psi_{3,2}\rangle$ form a string. They lie between the inhomogeneities v_2 and v_3. One of the roots coincides with v_1.

days of the planar $\mathcal{N} = 4$ SYM spectral problem [163]. There exist different approaches to treat theses solutions properly, cf. [161]. Some of them might also be applicable for the inhomogeneous spin chain with mixed representations that is associated with the Yangian invariant $|\Psi_{3,1}\rangle$.

The second three-site invariant discussed in section 2.4.1.4 is $|\Psi_{3,2}\rangle$. Its monodromy is characterized by, cf. (2.70) and (2.71),

$$\begin{gathered}\mathcal{V}_1 = \bar{\mathcal{D}}_{c_1}\,, \quad \mathcal{V}_2 = \bar{\mathcal{D}}_{c_2}\,, \quad \mathcal{V}_3 = \mathcal{D}_{c_3}\,,\\ v_1 = v_3 - 1 + c_1\,, \quad v_2 = v_3 - 1 - c_3\,, \quad c_1 + c_2 + c_3 = 0\,,\end{gathered} \tag{3.50}$$

where $c_1, c_2 \in -\mathbb{N}$. With the trivial normalization (2.72) of this monodromy and the form of the highest weights in (2.43), this turns (3.18) into

$$\alpha(u) = \frac{u - v_3 + c_3}{u - v_3}\,, \quad \delta(u) = \frac{u - v_3 + 1}{u - v_3 + 1 + c_3}\,. \tag{3.51}$$

Evidently, these functions solve the relation (3.36). The other relation (3.37) is then uniquely solved by

$$Q(u) = \frac{\Gamma(u - v_3 + c_3 + 1)}{\Gamma(u - v_3 + 1)} = \prod_{k=1}^{c_3}(u - v_3 + k)\,, \tag{3.52}$$

assuming we require the solution to be of the form (3.22). This yields the Bethe roots

$$u_k = v_3 - k \quad \text{for} \quad k = 1, \ldots, c_3\,. \tag{3.53}$$

Also for this sample solution they form a string, see figure 3.3. From the representations in (3.50) we work out the reference state (3.17) of the corresponding Bethe vector,

$$|\Omega\rangle = (\bar{\mathbf{a}}_2^1)^{-c_1}(\bar{\mathbf{a}}_2^2)^{-c_2}(\bar{\mathbf{a}}_1^3)^{c_3}|0\rangle\,. \tag{3.54}$$

The operator $B(u_{-c_1+1})$ diverges analogously to the one in the first three-site invariant because $u_{-c_1+1} = v_1$. Using the prescription from above we are led to a finite Bethe vector. By means of an explicit calculation for small absolute values of the representation labels we obtain

$$|\Psi\rangle = B(u_1)\cdots B(u_{c_3})|\Omega\rangle = (-1)^{c_3}(1 \bullet 3)^{-c_1}(2 \bullet 3)^{-c_2}|0\rangle \propto |\Psi_{3,2}\rangle\,. \tag{3.55}$$

This agrees with the three-site invariant $|\Psi_{3,2}\rangle$ from (2.73).

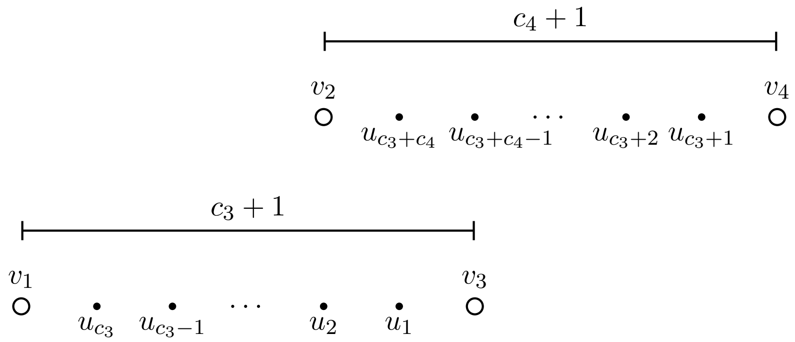


Figure 3.4: The Bethe roots utilized for the construction of the four-site Yangian invariant $|\Psi_{4,2}(z)\rangle$, i.e. to the R-matrix $R_{\mathcal{D}_{c_3}\mathcal{D}_{c_4}}(z)$, form two real strings in the complex plane. The difference between the endpoints of the two strings is the spectral parameter $z := v_3 - v_4$ of the R-matrix, cf. (2.81). The number of Bethe roots per string is given by the representation labels c_3 and c_4.

3.2.2.3 Four-Site Invariant and Superposition

In section 2.4.1.5 we investigated the four-site sample invariant $|\Psi_{4,2}(v_3 - v_4)\rangle$, which corresponds to an R-matrix. The associated monodromy matrix $M_{4,2}(u)$ is determined by, cf. (2.76) and (2.77),

$$\begin{gathered}\mathcal{V}_1 = \bar{\mathcal{D}}_{c_1}\,, \quad \mathcal{V}_2 = \bar{\mathcal{D}}_{c_2}\,, \quad \mathcal{V}_3 = \mathcal{D}_{c_3}\,, \quad \mathcal{V}_4 = \mathcal{D}_{c_4}\,,\\ v_1 = v_3 - 1 - c_3\,, \quad v_2 = v_4 - 1 - c_4\,, \quad c_1 + c_3 = 0\,, \quad c_2 + c_4 = 0\,.\end{gathered} \tag{3.56}$$

With the trivial normalization (2.78) of the monodromy and the highest weights (2.43) of the representations, the eigenvalues (3.18) become

$$\alpha(u) = \frac{u - v_3 + c_3}{u - v_3}\,\frac{u - v_4 + c_4}{u - v_4}\,, \quad \delta(u) = \frac{u - v_3 + 1}{u - v_3 + 1 + c_3}\,\frac{u - v_4 + 1}{u - v_4 + 1 + c_4}\,. \tag{3.57}$$

They satisfy the functional relation (3.36). The other relation (3.37) is solved by

$$Q(u) = \frac{\Gamma(u - v_3 + c_3 + 1)}{\Gamma(u - v_3 + 1)}\,\frac{\Gamma(u - v_4 + c_4 + 1)}{\Gamma(u - v_4 + 1)} = \prod_{k=1}^{c_3}(u - v_3 + k)\prod_{k=1}^{c_4}(u - v_4 + k)\,. \tag{3.58}$$

Assuming this Q-function to be of the form (3.22) assures the uniqueness of this solution. The zeros of this Q-function yield the Bethe roots

$$\begin{aligned} u_k &= v_3 - k \quad \text{for} \quad k = 1, \ldots, c_3\,,\\ u_{k+c_3} &= v_4 - k \quad \text{for} \quad k = 1, \ldots, c_4\,. \end{aligned} \tag{3.59}$$

They arrange into two strings, see figure 3.4. The Bethe vector (3.19) with these roots is constructed from the reference state (3.17)

$$|\Omega\rangle = (\bar{\mathbf{a}}_2^1)^{c_3}(\bar{\mathbf{a}}_1^3)^{c_3}(\bar{\mathbf{a}}_2^2)^{c_4}(\bar{\mathbf{a}}_1^4)^{c_4}|0\rangle\,, \tag{3.60}$$

which is fixed by the representations in (3.56). The manual evaluation of (3.19) for small values of c_3 and c_4 leads to

$$\begin{aligned}|\Psi\rangle &= B(u_1)\cdots B(u_{c_3})B(u_{c_3+1})\cdots B(u_{c_3+c_4})|\Omega\rangle\\ &= (-1)^{c_3+c_4}c_3!c_4!\prod_{l=1}^{\min(c_3,c_4)}(v_3-v_4+c_4-l+1)^{-1}\sum_{k=0}^{\min(c_3,c_4)}\frac{1}{(c_3-k)!(c_4-k)!k!}\\ &\quad\cdot\prod_{l=k+1}^{\min(c_3,c_4)}(v_3-v_4-c_3+l)\,(1\bullet 3)^{c_3-k}(2\bullet 4)^{c_4-k}(2\bullet 3)^k(1\bullet 4)^k|0\rangle\\ &\propto |\Psi_{4,2}(v_3-v_4)\rangle\,.\end{aligned} \tag{3.61}$$

This matches the Yangian invariant $|\Psi_{4,2}(z)\rangle$ from (2.79) with (2.80), (2.81) and (2.83). Consequently, we can interpret the R-matrix $R_{\mathcal{D}_{c_3}\mathcal{D}_{c_4}}(z)$, which is equivalent to this Yangian invariant, as a special Bethe vector.

Let us conclude the investigation of sample invariants with a comment on the general structure of the set of solutions to the functional relations (3.36) and (3.37). We observe that the solution of these relations defined by (3.57) and (3.58) is in fact the product of two two-site solutions, which we derived in section 3.2.2.1. There is a general principle behind this simple observation. Provided two solutions $(\alpha_1(u),\delta_1(u),Q_1(u))$ and $(\alpha_2(u),\delta_2(u),Q_2(u))$ of the functional relations, the product

$$(\alpha_1(u)\alpha_2(u),\delta_1(u)\delta_2(u),Q_1(u)Q_2(u)) \tag{3.62}$$

forms a new solution of these relations. Therefore we can obtain new Yangian invariants by "superposing" known ones. For instance, it should also be possible to combine a two-site solution and one of the three-site solutions of section 3.2.2.2 with this method.

3.2.3 Classification of Solutions

After studying sample solutions of the functional relations (3.36) and (3.37) in the previous section, we review a classification of the solutions of (3.36) found in [3].[2] As already mentioned before and experienced for the sample solutions, this equation is the crucial part of the functional relations because it constrains the monodromy matrix. Given a solution, the remaining first order difference equation (3.37) can typically be solved uniquely without any problems. Therefore a classification of the solutions of (3.36) should be thought of as a classification of all compact invariants of the Yangian of $\mathfrak{gl}(2)$. With mild restrictions on the form of the monodromy, one finds that each solution of (3.36) corresponds to a permutation and vice versa.

Let us begin by explaining the class of monodromies employed in this section. We choose the quantum space to be a tensor product of the two types of oscillator representations of section 2.4.1.1. The sites with "dual" representations are placed left of those with "ordinary" ones,

$$\mathcal{V}_1\otimes\cdots\otimes\mathcal{V}_N=\bar{\mathcal{D}}_{c_1}\otimes\cdots\otimes\bar{\mathcal{D}}_{c_K}\otimes\mathcal{D}_{c_{K+1}}\otimes\cdots\otimes\mathcal{D}_{c_N}\,. \tag{3.63}$$

We used this setup already for all the sample invariants in section 2.4.1, which we just reexamined in section 3.2.2. Here we impose in addition that the normalization of the

[2]The publication [3] is co-authored by the creator of this thesis. However, the results discussed in the present section were obtained before he joined that project. We review them in this dissertation because they provide important structural insights into the Bethe ansatz for Yangian invariants.

Lax operators (2.18) contained in the monodromy is trivial, $f_{\mathcal{V}_i} = 1$. This condition is compatible with the sample invariants because we found by hindsight that the overall normalization of the monodromy is trivial for all of them. To proceed, we compute the eigenvalues $\alpha(u)$ and $\delta(u)$ in (3.18) for this class of monodromies using the highest weights (2.43). Inserting the result into the functional relation (3.36) provides us with the explicit form of the equation we want to investigate,

$$\prod_{i=K+1}^{N} \frac{u - v_i + c_i}{u - v_i} \prod_{i=1}^{K} \frac{u - 1 - v_i + c_i}{u - 1 - v_i} = 1\,. \tag{3.64}$$

The solutions of this equation are most easily classified after changing variables to[3]

$$v_i' = v_i - \frac{c_i}{2} + \begin{cases} 1 & \text{for} \quad i = 1, \dots, K\,, \\ 0 & \text{for} \quad i = K+1, \dots, N\,. \end{cases} \tag{3.65}$$

Furthermore, we introduce, cf. [108],

$$v_i^{\pm} = v_i' \pm \frac{c_i}{2}\,. \tag{3.66}$$

These definitions transform (3.64) into

$$\prod_{i=1}^{N} (u - v_i^+) = \prod_{i=1}^{N} (u - v_i^-)\,. \tag{3.67}$$

For these two N-th order polynomials to be equal, their roots have to agree. Thus the solutions of (3.67) are in one-to-one correspondence with permutations σ of N elements,

$$v_{\sigma(i)}^+ = v_i^- \tag{3.68}$$

for $i = 1, \dots, N$. Consequently, the solutions of (3.36) and therefore also the associated Yangian invariants are classified by permutations. Equation (3.68) imposes N constraints on the $2N$ parameters v_i and c_i of the monodromy that enters the Yangian invariance condition (2.24).

We remark that to recover the conditions on the monodromies of the sample invariants $|\Psi_{N,K}\rangle$ in section 3.2.2 from (3.68), we have to choose the permutation of N elements to be the cyclic shift $\sigma(i) = i + K$. For the invariants $|\Psi_{2,1}\rangle$, $|\Psi_{3,1}\rangle$, $|\Psi_{3,2}\rangle$ and $|\Psi_{4,2}\rangle$ this choice turns (3.68) into (3.38), (3.44), (3.50) and (3.56), respectively.

Even though we derived the key condition (3.68) within the context of the Bethe ansatz, it is a property of the Yangian invariants themselves and not tied to the Bethe ansatz construction. In fact, we already encountered such a condition in the introductory section 1.3.6 on Yangian invariant deformations of SYM scattering amplitudes, cf. (1.56). It will appear again in chapter 4 during the construction of Yangian invariants with oscillator representations of $\mathfrak{u}(p, q|m)$ using Graßmannian matrix models. Hence it is not tied to the compact bosonic algebra $\mathfrak{u}(2)$ either. Probably the simplicity of (3.68) is, at least in part, related to the restricted class of oscillator representations that we consider in most parts of this thesis.

[3]This equation for v_i' differs from the corresponding equation (40) in [3] by a shift of 1 at the dual sites. This shift originates from a shift of the inhomogeneities of the Lax operators at those sites.

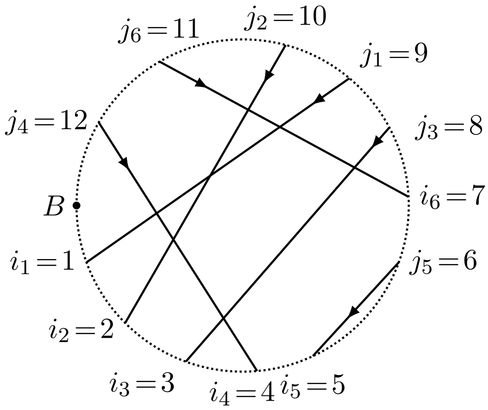


Figure 3.5: Example of a Baxter lattice containing $L = 6$ lines that are specified in terms of their endpoints by $\mathbf{G} = ((1,9),(2,10),(3,8),(4,12),(5,6),(7,11))$. The k-th line has the endpoints (i_k, j_k). An arrow defines its orientation and we assign to it a rapidity θ_k, which is not displayed in this figure.

3.3 Six-Vertex Model

Let us shift gears for a moment and discuss the *six-vertex model.* It is a prime example of an exactly solvable model in two-dimensional statistical mechanics, see e.g. the classic monograph [21]. Typically, this model is studied on a square lattice with periodic boundary conditions. In this setting the exact expression for the partition function is well-known for lattices of finite size [164, 165, 166]. However, it is given in an implicit form requiring the solutions of certain Bethe equations. The model has also been investigated on more general planar lattices [167], so-called *Baxter lattices*, which are in general non-rectangular. It is probably less known that on these lattices the partition function for fixed boundary conditions can be computed exactly using Baxter's *perimeter Bethe ansatz* [168]. In this approach the partition function is identified with a Bethe wave function. Astonishingly, the solutions of the Bethe equations are known explicitly, which is in contrast to most other applications of the Bethe ansatz. Of course, we already encountered another of these rare situations, the Bethe ansatz for Yangian invariants of section 3.2. This similarity is not a coincidence. In section 3.4 we will explain that the Bethe ansatz for Yangian invariants can be understood as a generalization of the perimeter Bethe ansatz.

Before we are able to establish that connection, we have to review Baxter's work [168]. We restrict our discussion to the rational limit of the six-vertex model. In this limit the model exhibits a $\mathfrak{su}(2)$ spin $\frac{1}{2}$ symmetry. Furthermore, our notation differs considerably from that in his publication. In section 3.3.1 we define the model on a Baxter lattice. Section 3.3.2 contains its solution in terms of the perimeter Bethe ansatz. We do not include Baxter's proof of this solution here because the connection with the Bethe ansatz for Yangian invariants in section 3.4 below will provide an alternative proof.

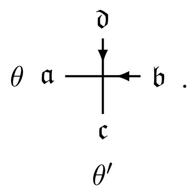
3.3.1 Rational Model on Baxter Lattices

A *Baxter lattice* consists of L straight lines that are placed arbitrarily in the interior of a circle in such a way that their endpoints lie on its perimeter. Each line is divided into edges by the points of intersection with the other lines. At most two lines are allowed to intersect at a point. An example of such a lattice is displayed in figure 3.5, where the perimeter is represented by a dotted circle. The L lines and their $2L$ endpoints are labeled counterclockwise starting at a reference point B on the perimeter. The k-th line has the endpoints (i_k, j_k) with $1 \leq i_k < j_k \leq 2L$. Its orientation is indicated by an arrow pointing from j_k to i_k. Furthermore, we allocate a complex rapidity θ_k to the line. Thus a Baxter lattice is determined by the ordered sets

$$\mathbf{G} = ((i_1, j_1), \ldots, (i_L, j_L))\,, \quad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_L)\,. \tag{3.69}$$

Each intersection of two lines defines a vertex to which we assign a Boltzmann weight, that is a matrix element of an R-matrix. Such a weight is a function of the rapidities of the lines and depends on states labeled by Gothic letters which are assigned to the adjacent edges,

$$\langle \mathfrak{a}, \mathfrak{c} | \tilde{R}_{\Box\Box'}(\theta - \theta') | \mathfrak{b}, \mathfrak{d} \rangle = \tag{3.70}$$



For the rational limit of the six-vertex model these weights are elements of the R-matrix

$$\tilde{R}_{\Box\Box'}(\theta - \theta') = \frac{1}{\theta - \theta' + 1} \begin{pmatrix} \theta - \theta' + 1 & 0 & 0 & 0 \\ 0 & \theta - \theta' & 1 & 0 \\ 0 & 1 & \theta - \theta' & 0 \\ 0 & 0 & 0 & \theta - \theta' + 1 \end{pmatrix}. \tag{3.71}$$

Apart from a change of the normalization, which we indicate by the tilde, this is the $\mathfrak{gl}(2)$ version of the R-matrix (2.10) used in the definition of the Yangian. The Gothic indices $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ take the values 1 or 2 denoting the states states $|1\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ or $|2\rangle = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, respectively. The R-matrix acts on the tensor product $|\mathfrak{b}, \mathfrak{d}\rangle := |\mathfrak{b}\rangle \otimes |\mathfrak{d}\rangle$ and its matrix elements are built with the bras $\langle \mathfrak{a}, \mathfrak{c}| := \langle \mathfrak{a}| \otimes \langle \mathfrak{c}|$. The six non-zero matrix elements of (3.71) correspond to vertex configurations with an equal number of incoming states $|1\rangle$, $|2\rangle$ and outgoing states $\langle 1|$, $\langle 2|$, respectively. This "conservation law" is referred to as *ice rule*.

The boundary conditions of the vertex model are specified by fixing the states at the boundary edges of the Baxter lattice. This is done by assigning state labels $\mathfrak{a}_{i_k}$ and $\mathfrak{a}_{j_k}$ with values 1 or 2 to the endpoints (i_k, j_k) of the lines, see figure 3.6. These labels are collectively denoted by

$$\boldsymbol{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_{2L})\,. \tag{3.72}$$

The partition function of the vertex model is

$$\mathcal{Z}(\mathbf{G}, \boldsymbol{\theta}, \boldsymbol{\mathfrak{a}}) = \sum_{\substack{\text{internal} \\ \text{state} \\ \text{config.}}} \prod_{\text{vertices}} \text{Boltzmann weight}\,, \tag{3.73}$$

where we sum over all possible state configurations of the internal edges. An additional prescription is necessary for lines which do not contain any vertex, see e.g. the line with endpoints $(5, 6)$ in figure 3.6. Such a line k contributes a factor of 1 to the partition

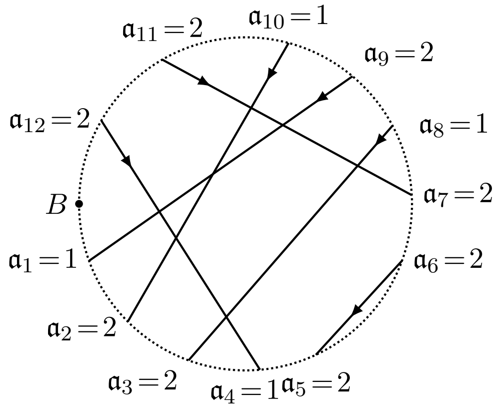


Figure 3.6: The sample Baxter lattice of figure 3.5 including boundary conditions, which consist of state labels in $\boldsymbol{\mathfrak{a}}$ that are assigned to the endpoints. Notice that the ice rule (3.74) is obeyed because the number of endpoints i_k with a state label $\mathfrak{a}_{i_k} = 1$ and that of endpoints j_k with $\mathfrak{a}_{j_k} = 1$ coincides. Thus $\boldsymbol{\mathfrak{a}}$ gives rise to the magnon positions $\mathbf{x} = (1, 4, 6, 9, 11, 12)$ via (3.80). These are instrumental in expressing the partition function $\mathcal{Z}(\mathbf{G}, \boldsymbol{\theta}, \boldsymbol{\mathfrak{a}})$ in terms of a Bethe wave function $\Phi(\mathbf{w}, \mathbf{u}, \mathbf{x})$ in (3.82).

functions if the state labels at the endpoints are identical, $\mathfrak{a}_{i_k} = \mathfrak{a}_{j_k}$. It contributes a factor of 0 if they differ, $\mathfrak{a}_{i_k} \neq \mathfrak{a}_{j_k}$. Thus the entire partition function vanishes in the latter case.

The ice rule for each vertex implies at a global level that for the partition function to be non-zero, the number of endpoints i_k at outward pointing boundary edges with $\mathfrak{a}_{i_k} = 1$ must be equal to that of endpoints j_k at inward pointing edges with $\mathfrak{a}_{j_k} = 1$,

$$\left|\{i_k \,|\, \mathfrak{a}_{i_k} = 1\}\right| = \left|\{j_k \,|\, \mathfrak{a}_{j_k} = 1\}\right| . \tag{3.74}$$

The analogous condition must be fulfilled for endpoints with the state label 2.

The R-matrix (3.71), which contains the Boltzmann weights of the vertex model, solves a Yang-Baxter equation, cf. section 2.1. This equation implies that the partition function does not change if a line is moved through a vertex without changing the order of endpoints at the perimeter. This property of the partition function is called Z-invariance.

3.3.2 Perimeter Bethe Ansatz Solution

The partition function (3.73) was computed exactly by Baxter in [168] by identifying it with a Bethe wave function. Such wave functions were introduced by Bethe in his original solution of the Heisenberg model [158]. Pedagogical accounts on that coordinate Bethe ansatz may be found in [169, 13]. It presents an alternative to the algebraic Bethe ansatz, which we recapitulated in section 3.1. To reproduce Baxter's result we need an extension of the coordinate Bethe ansatz to inhomogeneous spin chains [170, 171].

In case of a chain with N sites and P magnon excitations the Bethe wave function is parametrized by

$$\mathbf{w} = (w_1, \ldots, w_N) , \quad \mathbf{u} = (u_1, \ldots, u_P) , \quad \mathbf{x} = (x_1, \ldots, x_P) , \tag{3.75}$$

which denote the inhomogeneities, the Bethe roots and the magnon positions satisfying $1 \le x_1 < \ldots < x_P \le N$, respectively. The wave function is given by

$$\Phi(\mathbf{w},\mathbf{u},\mathbf{x}) = \sum_{\rho} \mathrm{A}(u_{\rho(1)},\ldots,u_{\rho(P)}) \prod_{k=1}^{P} \phi_{x_k}(u_{\rho(k)},\mathbf{w})\,, \tag{3.76}$$

where the summation runs over all permutations ρ of P elements. Furthermore, the factor

$$\mathrm{A}(u_{\rho(1)},\ldots,u_{\rho(P)}) = \prod_{1\le k<l\le P} \frac{u_{\rho(k)}-u_{\rho(l)}+1}{u_{\rho(k)}-u_{\rho(l)}} \tag{3.77}$$

is independent of the inhomogeneities and

$$\phi_x(u,\mathbf{w}) = \prod_{j=1}^{x-1}(u-w_j+1) \prod_{j=x+1}^{N}(u-w_j) \tag{3.78}$$

is a single particle wave function, see also [134].[4] The Bethe equations

$$\prod_{i=1}^{N} \frac{u_k-w_i+1}{u_k-w_i} = -\prod_{l=1}^{P} \frac{u_k-u_l+1}{u_k-u_l-1} \tag{3.79}$$

with $1 \le k \le P$ are obtained by imposing periodicity of (3.76) in the magnon positions. These equations ensure that the wave functions (3.76) for different magnon configurations $\mathbf{x}$ are components of the transfer matrix eigenvectors of the inhomogeneous Heisenberg spin chain. For generic Bethe roots $\mathbf{u}$, (3.76) is often called “off-shell” Bethe wave function. It becomes “on-shell” once Bethe roots obeying (3.79) are inserted.

Next, we will identify the partition function (3.73) with the Bethe wave function (3.76). We restrict to lattice configurations for which the ice rule (3.74) holds because otherwise the partition function vanishes. To perform the identification we proceed as follows, where in particular the parameters $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{x}$ of (3.76) are related to the variables $\mathbf{G}$, $\boldsymbol{\theta}$ and $\boldsymbol{\mathfrak{a}}$ of (3.73):

1. For a Baxter lattice consisting of L lines, we choose a wave function of a spin chain with $N = 2L$ sites and $P = L$ excitations. Such a configuration is referred to as “half-filling”.

2. The lattice configuration in $\boldsymbol{\mathfrak{a}}$ and $\mathbf{G}$ determines the magnon positions $\mathbf{x}$. They are defined as the endpoint positions i_k at outward pointing edges with $\mathfrak{a}_{i_k} = 1$ and j_k at edges directed inwards with $\mathfrak{a}_{j_k} = 2$,

$$\{x_k\} = \{i_k | \mathfrak{a}_{i_k}\!=\!1\} \cup \{j_k | \mathfrak{a}_{j_k}\!=\!2\}\,. \tag{3.80}$$

   These positions are then ordered by imposing $1 \le x_1 < \ldots < x_L \le 2L$. An example is provided in figure 3.6.

3. $\mathbf{G}$ and the rapidities $\boldsymbol{\theta}$ fix the inhomogeneities $\mathbf{w}$ and the Bethe roots $\mathbf{u}$. For each line k with endpoints (i_k, j_k) we define

$$w_{i_k} = \theta_k + 1\,, \quad w_{j_k} = \theta_k + 2\,, \quad u_k = \theta_k + 1\,. \tag{3.81}$$

   Perhaps surprisingly, this constitutes an exact solution of the Bethe equations (3.79). This claim is most easily verified after writing the Bethe equations in polynomial form to circumvent divergencies, see also the comment below (3.24).

[4] For a homogeneous spin chain with $w_j = 0$, the Bethe wave function (3.76) takes a more common form after dividing (3.76) by $\mathrm{A}(u_1,\ldots,u_P)$ to obtain the S-matrix and changing variables to $p_k = -i \log \frac{u_k+1}{u_k}$.

After following these steps, the partition function (3.73) is given by the Bethe wave function (3.76),

$$\mathcal{Z}(\mathbf{G}, \boldsymbol{\theta}, \mathfrak{a}) = \mathcal{C}(\mathbf{G}, \boldsymbol{\theta})^{-1}(-1)^{\mathcal{K}(\mathbf{G},\mathfrak{a})}\Phi(\mathbf{w}, \mathbf{u}, \mathbf{x})\,. \tag{3.82}$$

Here $\mathcal{K}(\mathbf{G}, \mathfrak{a})$ denotes the number of endpoints i_k with state label $\mathfrak{a}_{i_k} = 2$,

$$\mathcal{K}(\mathbf{G}, \mathfrak{a}) = \left|\{i_k \,|\, \mathfrak{a}_{i_k} = 2\}\right|\,. \tag{3.83}$$

The normalization is independent of $\mathfrak{a}$ and reads

$$\mathcal{C}(\mathbf{G}, \boldsymbol{\theta}) = \Phi(\mathbf{w}, \mathbf{u}, \mathbf{x}_0)\,. \tag{3.84}$$

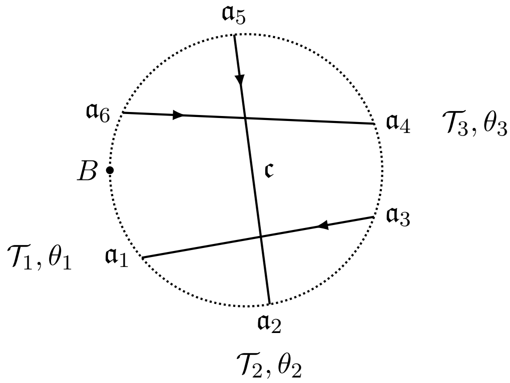
Here $\mathbf{x}_0 = (i_1, \ldots, i_L)$ is computed from (3.80) with the boundary conditions $\mathfrak{a}_0 = (1, \ldots, 1)$. The explicit expression (3.82) for the partition function is the *perimeter Bethe ansatz* solution of the six-vertex model on a Baxter lattice in the rational limit [168]. As mentioned earlier, we refrain from presenting the original proof here. Instead, we will prove this expression in section 3.4.4 by showing it to be a special case of the Bethe ansatz for Yangian invariants.

3.4 From Vertex Models to Yangian Invariance

We already alluded to a relation between the rational six-vertex model and Yangian invariance in the introduction of the previous section 3.3. Here we expose the details of this connection. As it turns out, it is not limited to the rational six-vertex model but extends to more general vertex models on Baxter lattices, which we define in section 3.4.1. In these models the symmetry algebra is generalized from $\mathfrak{gl}(2)$ to $\mathfrak{gl}(n)$. Furthermore, the lines of the lattice can carry representations that are different from the defining representation $\square$. In particular, we may use the compact oscillator representations of section 2.4.1.1. In section 3.4.2 we show that the partition functions of these vertex models are components of Yangian invariant vectors. Consequently, the Bethe ansatz for Yangian invariants of section 3.2 is applicable for the construction of these partition functions in the $\mathfrak{gl}(2)$ case, as detailed in section 3.4.3. Finally, in section 3.4.4 we demonstrate explicitly that this Bethe ansatz reduces to the perimeter Bethe ansatz reviewed in section 3.3.2 if we restrict the representations of the lines to get back to the rational six-vertex model.

3.4.1 Vertex Models on Baxter Lattices

We generalize the vertex model of section 3.3.1, where all lines of the Baxter lattice carry the defining representation $\square$ of $\mathfrak{gl}(2)$, to a class of models for which each line is associated with a, from the outset, different representation of $\mathfrak{gl}(n)$. Let us reiterate the definition of the Baxter lattice for such models. A sample lattice is provided in the left part of figure 3.7. The construction of the Baxter lattice makes use of a dotted circle on which we mark a reference point B. However, both, the circle and the reference point, are not part of the lattice itself. We choose L straight lines whose endpoints lie on the dotted circle. Moreover, we demand that at most two lines intersect at a point in the interior of the circle. The L lines and their $2L$ endpoints are labeled counterclockwise starting at the reference point. The two endpoints of the k-th line are $i_k < j_k$. An arrow pointing from j_k to i_k provides an orientation of the line. Notice that the orientation clearly depends on the position of the reference point. Furthermore, the k-th line possesses a spectral parameter θ_k and it carries a representation $\mathcal{T}_k$ of $\mathfrak{gl}(n)$. Recall that our terminology does not distinguish between a



$$\begin{aligned}&\mathcal{Z}(\mathbf{G}, \boldsymbol{\mathcal{T}}, \boldsymbol{\theta}, \boldsymbol{\mathfrak{a}}) \\ &= \sum_{\mathfrak{c}} \langle \mathfrak{a}_1, \mathfrak{a}_2 | R_{\mathcal{T}_1\,\mathcal{T}_2}(\theta_1 - \theta_2) | \mathfrak{a}_3, \mathfrak{c}\rangle \\ &\quad \cdot \langle \mathfrak{c}, \mathfrak{a}_4 | R_{\mathcal{T}_2\,\mathcal{T}_3}(\theta_2 - \theta_3) | \mathfrak{a}_5, \mathfrak{a}_6\rangle\end{aligned}$$

Figure 3.7: The left side shows a sample (generalized) Baxter lattice with $L = 3$ lines whose endpoints are encoded in $\mathbf{G} = ((1,3),(2,5),(4,6))$. The k-th line carries a $\mathfrak{gl}(n)$ representation $\mathcal{T}_k$, a spectral parameter θ_k, two state labels $\mathfrak{a}_{i_k}$ and $\mathfrak{a}_{j_k}$ at the endpoints (i_k, j_k) and an arrow indicating its orientation. The dotted circle and the reference point B are not part of the lattice. On the left we present the associated partition function $\mathcal{Z}(\mathbf{G}, \boldsymbol{\mathcal{T}}, \boldsymbol{\theta}, \boldsymbol{\mathfrak{a}})$.

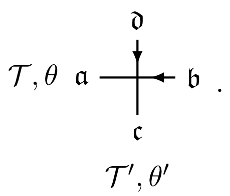
representation and the associated vector space, as mentioned after (2.4). We assign basis states of $\mathcal{T}_k$ labeled by $\mathfrak{a}_{i_k}$ and $\mathfrak{a}_{j_k}$ to the endpoints of the line. These Gothic state labels may take the values $1, 2, \ldots, \dim(\mathcal{T}_k)$. In summary, a (generalized) *Baxter lattice* including boundary conditions is defined by the ordered sets

$$\begin{gathered}\mathbf{G} = ((i_1, j_1), \ldots, (i_L, j_L))\,, \\ \boldsymbol{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_L)\,, \quad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_L)\,, \quad \boldsymbol{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_{2L})\,.\end{gathered} \tag{3.85}$$

Let us remark that we may choose the $\mathcal{T}_k$ out of the classes of oscillator representations introduced in section 2.3, yet this restriction is not necessary here.[5] However, we will have to impose some conditions on the $\mathcal{T}_k$ later on in section 3.4.2.

To define a vertex model on this type of Baxter lattice, we generalize the Boltzmann weights of section 3.3.1 to

$$\langle \mathfrak{a}, \mathfrak{c} | R_{\mathcal{T}\,\mathcal{T}'}(\theta - \theta') | \mathfrak{b}, \mathfrak{d}\rangle = \tag{3.86}$$

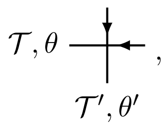


These weights are computed with respect to orthonormal bases of the $\mathfrak{gl}(n)$ representations $\mathcal{T}$ and $\mathcal{T}'$ whose states are labeled by the Gothic indices $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, $\mathfrak{d}$, respectively. They

[5]The Gothic letters used here as state labels coincide with the Greek oscillator indices of section 2.3 for the $\mathfrak{u}(n)$ representations $\mathcal{D}_1$ and $\bar{\mathcal{D}}_{-1}$ because these are spanned by the states $\bar{\mathbf{a}}_\alpha|0\rangle$ with a single Greek index $\alpha = 1, \ldots, n$.

are elements of the R-matrix

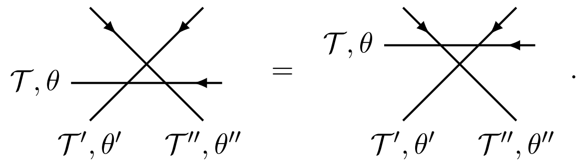
$$R_{\mathcal{T}\mathcal{T}'}(\theta-\theta') = \quad (3.87)$$



which acts on the tensor product $\mathcal{T}\otimes\mathcal{T}'$ and is a function of the spectral parameters θ and θ'. We briefly recall the graphical notation for such R-matrices from section 2.1. Each line is associated with one representation space. The arrow on the line defines the order in which multiple R-matrices act on that space. An R-matrix "earlier" on the line acts before one appearing "later" on the line. Thus the arrows are directed from the "inputs" of the R-matrix towards the "outputs". In the component language (3.86) we may think of them as pointing from the ket to the bra states. In the following we will use the operator and the component language interchangeably. The R-matrix (3.87) is a solution of the Yang-Baxter equation

$$\begin{aligned}&R_{\mathcal{T}\mathcal{T}'}(\theta-\theta')R_{\mathcal{T}\mathcal{T}''}(\theta-\theta'')R_{\mathcal{T}'\mathcal{T}''}(\theta'-\theta'')\\&=R_{\mathcal{T}'\mathcal{T}''}(\theta'-\theta'')R_{\mathcal{T}\mathcal{T}''}(\theta-\theta'')R_{\mathcal{T}\mathcal{T}'}(\theta-\theta')\end{aligned} \tag{3.88}$$

acting in the tensor product $\mathcal{T}\otimes\mathcal{T}'\otimes\mathcal{T}''$. Employing the graphical notation this becomes



(3.89)

These equations are the analogues of (2.11) and (2.12) for the general class of $\mathfrak{gl}(n)$ representations considered here.

This setup allows us to define the partition function of a vertex model on a (generalized) Baxter lattice following (3.73) as

$$\mathcal{Z}(\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\boldsymbol{\mathfrak{a}}) = \sum_{\substack{\text{internal}\\\text{state}\\\text{config.}}}\prod_{\text{vertices}}\text{Boltzmann weight}\,, \tag{3.90}$$

see once more the example in figure 3.7. The sum in this formula runs over all possible state configurations at the internal edges of the lattice. For an internal edge belonging to a line with a representation $\mathcal{T}$ these are the basis states of that representation. The product is over all vertices of the lattice. Each vertex is associated with a Boltzmann weight of the form (3.86). The states at the boundary edges of the lattice are fixed by the state labels in $\boldsymbol{\mathfrak{a}}$. In case a line consists of a single edge and the two state labels assigned to it by $\boldsymbol{\mathfrak{a}}$ differ, the partition function vanishes.

After defining the partition function in (3.90) in component language using the Boltzmann weights in (3.86), we rephrase it using directly the R-matrix (3.87). In such an operator language the partition function is a matrix element of a certain product of R-matrices. The R-matrices appearing in this product as well as their order are prescribed by the form of the Baxter lattice. The matrix multiplications in this product correspond to the sum and the product in (3.90). A non-intersecting line in the Baxter lattice becomes an identity operator on the corresponding representation space. The matrix element of the product of R-matrices is selected by the boundary conditions in $\boldsymbol{\mathfrak{a}}$.

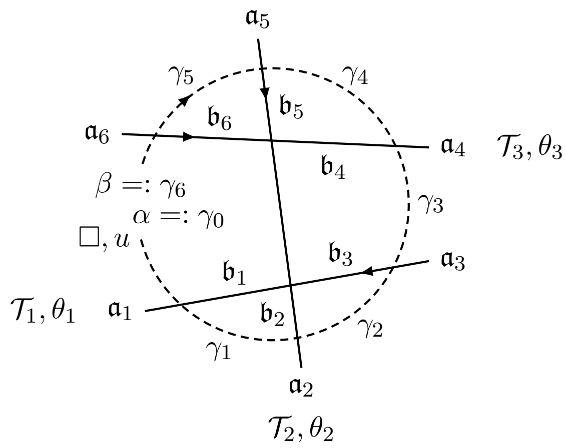
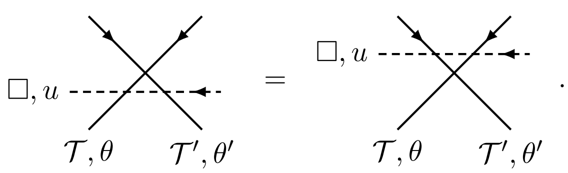


Figure 3.8: The Baxter lattice introduced in the example of figure 3.7 after the dotted circle has been replaced by a dashed auxiliary space line in the defining representation $\square$ with a spectral parameter u and states labeled α, β at the endpoints. The indices γ_i are assigned to the edges of this auxiliary space. The states at the edges connecting this space with the Baxter lattice are labeled $\mathfrak{b}_i$.

3.4.2 Partition Function as Yangian Invariant

To understand the relation between the partition function $\mathcal{Z}(\mathbf{G}, \boldsymbol{\mathcal{T}}, \boldsymbol{\theta}, \boldsymbol{\mathfrak{a}})$ on a Baxter lattice and a Yangian invariant $|\Psi\rangle$, we start out by deriving an identity obeyed by this partition function. In a second step, we then reformulate this identity in the QISM language to show that it is identical to the Yangian invariance condition (2.24).

In order to derive said identity, we replace the dotted circle, which appeared in the construction of the Baxter lattice, by a dashed arc that is opened at the reference point B. Furthermore, we extend the lines of the Baxter lattice such that they intersect the arc. See the example lattice in figure 3.8. The dashed arc now represents an actual space. This auxiliary space carries the defining representation $\square = \mathbb{C}^n$ of $\mathfrak{gl}(n)$ and a spectral parameter u. It is oriented counterclockwise. The bra and ket states at its endpoints are labeled by α and β, respectively, which can take the values $1, \ldots, n$. Each line of the Baxter lattice intersects the arc twice. This creates an additional layer of vertices at the boundary of the lattice. The Boltzmann weights associated with these vertices are elements of R-matrices of the type $R_{\square\mathcal{T}}(u - \theta)$ or $R_{\mathcal{T}\square}(\theta - u)$. These R-matrices are referred to as Lax operators, cf. section 2.1, and they satisfy Yang-Baxter equations like



(3.91)

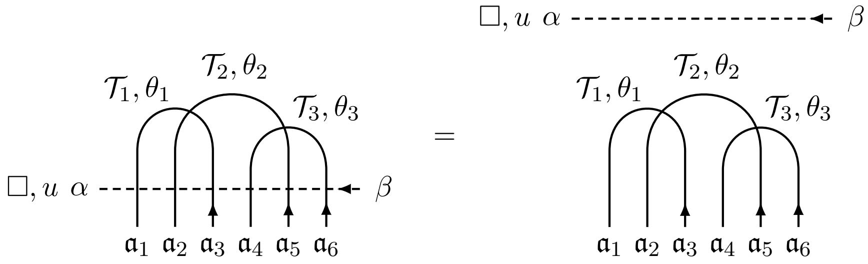
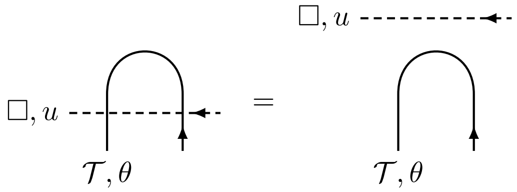


Figure 3.9: Disentangling the dashed auxiliary line from the solid lines by means of (3.91) and (3.92) results in a non-trivial identity for $\mathcal{Z}(\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\mathfrak{a})$ of the sample Baxter lattice in figure 3.7. We deformed the lattice to highlight the horizontal row of vertices involving the auxiliary line which will be expressed as a spin chain monodromy shortly, cf. figure 3.10.

Furthermore, we impose the unitarity condition

$$R_{\Box\mathcal{T}}(u-\theta)R_{\mathcal{T}\Box}(\theta-u)=1\,,\quad \text{i.e.} \tag{3.92}$$



employing the graphical notation. We encountered such a condition already for the Lax operators of oscillator representations in section 2.4.1.1. The Yang-Baxter equation (3.91) and the unitarity condition (3.92) allow us to disentangle the dashed auxiliary space line from the L solid lines constituting the Baxter lattice. As shown for an example in figure 3.9, this yields an identity for the partition function $\mathcal{Z}(\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\mathfrak{a})$. To derive this identity for a general Baxter lattice, let us denote the Boltzmann weights involving the auxiliary space by

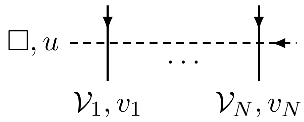
$$\begin{aligned}
&\mathcal{M}_{\alpha\beta}(u,\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\mathfrak{a},\mathfrak{b})\\
&\quad=\sum_{\gamma_1,\ldots,\gamma_{2L-1}=1}^{n}\Bigg(\prod_{k=1}^{L}\langle\gamma_{i_k-1},\mathfrak{a}_{i_k}|R_{\Box\mathcal{T}_k}(u-\theta_k)|\gamma_{i_k},\mathfrak{b}_{i_k}\rangle\\
&\qquad\qquad\cdot\langle\mathfrak{b}_{j_k},\gamma_{j_k-1}|R_{\mathcal{T}_k\Box}(\theta_k-u)|\mathfrak{a}_{j_k},\gamma_{j_k}\rangle\Bigg)_{\substack{\gamma_0:=\alpha\\ \gamma_{2L}:=\beta}}\,.
\end{aligned}\tag{3.93}$$

Here each of the L lines of the Baxter lattice leads to two Boltzmann weights in the product. Recall from the definition of the lattice around (3.85) that the endpoints of the k-th line are denoted $i_k<j_k$. Figure 3.8 provides an example for the assignment of the state labels γ_i and $\mathfrak{b}_i$ to the edges of the lattice. States at the edges of the auxiliary space line are labeled by γ_i with $i=0,\ldots,2L$. The state labels $\mathfrak{b}=(\mathfrak{b}_1,\ldots,\mathfrak{b}_{2L})$ are placed at those edges that connect the layer of vertices involving the auxiliary space with the Baxter lattice on which the partition function is defined. Eventually, equating the original entangled situation to the disentangled one results in the sought after identity

$$\sum_{\mathfrak{b}}\mathcal{M}_{\alpha\beta}(u,\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\mathfrak{a},\mathfrak{b})\,\mathcal{Z}(\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\mathfrak{b})=\delta_{\alpha\beta}\,\mathcal{Z}(\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\mathfrak{a})\,. \tag{3.94}$$

In this equation the $\delta_{\alpha\beta}$ on the r.h.s. corresponds to the unraveled auxiliary line. For a sample Baxter lattice we know this identity already from figure 3.9.

For the purpose of translating the identity (3.94) in the QISM formalism, we show that the combination of Boltzmann weights in $\mathcal{M}_{ab}(u, \mathbf{G}, \boldsymbol{\mathcal{T}}, \boldsymbol{\theta}, \mathfrak{a}, \mathfrak{b})$ can be interpreted as a matrix element of a monodromy $M(u)$. The monodromy of a spin chain with N sites is defined as in (2.19) by

$$M(u) = R_{\square\mathcal{V}_1}(u - v_1) \cdots R_{\square\mathcal{V}_N}(u - v_N) =$$


. (3.95)

To the j-th site we assign an inhomogeneity $v_j \in \mathbb{C}$ and a representation $\mathcal{V}_j$ of $\mathfrak{gl}(n)$. Hence the total quantum space of the monodromy is $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_N$. Its auxiliary space is in the defining representation $\square$ and the matrix elements with respect to this space are

$$M_{\alpha\beta}(u) := \langle\alpha|M(u)|\beta\rangle\,. \tag{3.96}$$

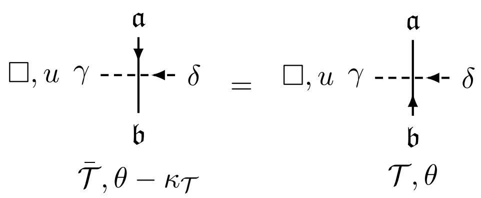
They are operators in the total quantum space. To proceed, we assume that the Lax operators belonging to the Boltzmann weights in (3.93) fulfill the crossing relation

$$R_{\square\bar{\mathcal{T}}}(u - \theta + \kappa_{\mathcal{T}}) = R_{\mathcal{T}\square}(\theta - u)^{\dagger}\,, \tag{3.97}$$

where the crossing parameter $\kappa_{\mathcal{T}}$ depends on the representation and the Hermitian conjugation acts solely on the representation space $\mathcal{T}$. The generators $\bar{J}_{\alpha\beta}$ of the dual representation $\bar{\mathcal{T}}$ appearing in (3.97) are defined in terms of the generators $J_{\alpha\beta}$ of $\mathcal{T}$ by

$$\bar{J}_{\alpha\beta} = -J_{\alpha\beta}^{\dagger}\,. \tag{3.98}$$

Recall from section 2.4.1.1 that the crossing relation (3.97) holds in particular in case of the compact oscillator representations discussed there. In what follows, we require the matrix elements $\langle\mathfrak{a}|J_{\alpha\beta}|\mathfrak{b}\rangle$ of the generators to be real. This leads to the component form of (3.97),

$$\begin{aligned}&\langle\gamma, \mathfrak{b}|R_{\square\bar{\mathcal{T}}}(u - \theta + \kappa_{\mathcal{T}})|\delta, \mathfrak{a}\rangle\\ &= \langle\mathfrak{a}, \gamma|R_{\mathcal{T}\square}(\theta - u)|\mathfrak{b}, \delta\rangle\,,\end{aligned}$$ i.e.


(3.99)

in the graphical notation. Applying (3.99) for the Boltzmann weights in the second line of (3.93) results in

$$\begin{aligned}&\mathcal{M}_{\alpha\beta}(u, \mathbf{G}, \boldsymbol{\mathcal{T}}, \boldsymbol{\theta}, \mathfrak{a}, \mathfrak{b})\\ &\quad= \sum_{\gamma_1,\dots,\gamma_{2L-1}=1}^{n} \Bigg(\prod_{k=1}^{L} \langle\gamma_{i_k-1}, \mathfrak{a}_{i_k}|R_{\square\mathcal{T}_k}(u - \theta_k)|\gamma_{i_k}, \mathfrak{b}_{i_k}\rangle\\ &\qquad\qquad \cdot \langle\gamma_{j_k-1}, \mathfrak{a}_{j_k}|R_{\square\bar{\mathcal{T}}_k}(u - \theta_k + \kappa_{\mathcal{T}_k})|\gamma_{j_k}, \mathfrak{b}_{j_k}\rangle \Bigg)_{\substack{\gamma_0:=\alpha\\ \gamma_{2L}:=\beta}}\,.\end{aligned} \tag{3.100}$$

From the index structure of this expression we infer that it is the matrix element of the monodromy (3.95) with $N = 2L$ sites,

$$\mathcal{M}_{\alpha\beta}(u, \mathbf{G}, \boldsymbol{\mathcal{T}}, \boldsymbol{\theta}, \mathfrak{a}, \mathfrak{b}) = \langle\mathfrak{a}|M_{\alpha\beta}(u)|\mathfrak{b}\rangle\,. \tag{3.101}$$

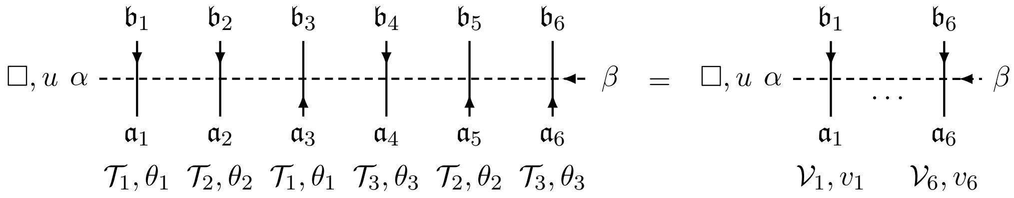


Figure 3.10: Rewriting of the summed Boltzmann weights in $\mathcal{M}_{\alpha\beta}(u, \mathbf{G}, \boldsymbol{\mathcal{T}}, \boldsymbol{\theta}, \boldsymbol{\mathfrak{a}}, \boldsymbol{\mathfrak{b}})$ on the l.h.s. as a matrix element $\langle \boldsymbol{\mathfrak{a}}|M_{\alpha\beta}(u)|\boldsymbol{\mathfrak{b}}\rangle$ of a monodromy on the r.h.s. for the example discussed in figure 3.9. After applying (3.99) to the l.h.s. all vertical lines have the same orientation. $\mathcal{V}_i$ and v_i of the resulting monodromy are given by (3.102) with $\mathbf{G}$ specified in the caption of figure 3.7.

See the example in figure 3.10. The labels $\mathbf{G}$, $\boldsymbol{\mathcal{T}}$, $\boldsymbol{\theta}$ on the l.h.s. of (3.101) encode the total quantum space of the monodromy. As is usual in the QISM, this information is hidden on the r.h.s. of the equation. We employed the notation $|\boldsymbol{\mathfrak{b}}\rangle := |\mathfrak{b}_1\rangle \otimes \cdots \otimes |\mathfrak{b}_{2L}\rangle \in \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_{2L}$. The k-th line of the Baxter lattice whose endpoints $i_k < j_k$ are defined in $\mathbf{G}$, cf. (3.85), determines two sites of the monodromy matrix with representations and inhomogeneities

$$\mathcal{V}_{i_k} = \mathcal{T}_k\,, \quad v_{i_k} = \theta_k \quad \text{and} \quad \mathcal{V}_{j_k} = \bar{\mathcal{T}}_k\,, \quad v_{j_k} = \theta_k - \kappa_{\mathcal{T}_k}\,. \tag{3.102}$$

What is more, we can associate a vector $|\Psi\rangle$ in the total quantum space with the partition function via the relation

$$\langle \boldsymbol{\mathfrak{a}}|\Psi\rangle := \mathcal{Z}(\mathbf{G}, \boldsymbol{\mathcal{T}}, \boldsymbol{\theta}, \boldsymbol{\mathfrak{a}})\,. \tag{3.103}$$

Making use of (3.101), (3.103) and the orthonormality relation for the states $|\boldsymbol{\mathfrak{b}}\rangle$, the identity (3.94) for the partition function becomes

$$\langle \boldsymbol{\mathfrak{a}}|M_{\alpha\beta}(u)|\Psi\rangle = \delta_{\alpha\beta}\langle \boldsymbol{\mathfrak{a}}|\Psi\rangle\,. \tag{3.104}$$

After dropping the bra state $\langle \boldsymbol{\mathfrak{a}}|$, this is precisely the component version (2.25) of the Yangian invariance condition in the QISM language. The Yangian invariant vector $|\Psi\rangle$ comprises the partition functions of a fixed Baxter lattice for all possible boundary conditions $\boldsymbol{\mathfrak{a}}$.

In this sense vertex models on Baxter lattices give rise to a special class of Yangian invariants with a monodromy defined by (3.102). The vertex model origin of these invariants allows us to interpret the associated intertwiners, cf. section 2.4.1.2, as products of R-matrices. The two-site sample invariant of section 2.4.1.3 with compact oscillator representations falls into this class. Its Baxter lattice consists of just a single line. The four-site invariant of section 2.4.1.5 may be understood as originating from a Baxter lattice with two intersecting lines. In contrast, the three-site Yangian invariants constructed in section 2.4.1.4 clearly leave the vertex model framework because their monodromies are not of the form (3.102). Notably, the vertex model interpretation established in this section only applies to Yangian invariants with an even number of sites.

3.4.3 Bethe Ansatz Solution

We argued in section 3.2 that Yangian invariants $|\Psi\rangle$ can be constructed using a Bethe ansatz and we showed this in detail for finite-dimensional $\mathfrak{gl}(2)$ representations. In the previous section we demonstrated that the partition functions of vertex models on Baxter lattices are encoded in a certain class of Yangian invariants. Here we present the Bethe ansatz solution corresponding to these invariants. We restrict to Baxter lattices with compact oscillator representations of $\mathfrak{gl}(2)$. This restriction also ensures that the two- and four-site sample solutions of the Bethe ansatz from section 3.2.2 are included in the following result as the special case of Baxter lattices with one and two lines, respectively.

Let us consider a Baxter lattice with L lines. The k-th line with endpoints $i_k < j_k$ and spectral parameter θ_k carries the compact oscillator representation $\mathcal{T}_k = \bar{\mathcal{D}}_{c_{i_k}}$ of $\mathfrak{gl}(2)$, see section 2.4.1.1. According to (3.102) the monodromy of the associated Yangian invariant has $N = 2L$ sites that are given by

$$\begin{gathered}\mathcal{V}_{i_k} = \bar{\mathcal{D}}_{c_{i_k}}\,, \quad \mathcal{V}_{j_k} = \mathcal{D}_{c_{j_k}}\,, \\ v_{i_k} = \theta_k\,, \quad v_{j_k} = \theta_k - c_{i_k} + 1\,, \quad c_{i_k} + c_{j_k} = 0\,.\end{gathered} \tag{3.105}$$

With this data the monodromy eigenvalues (3.18) read

$$\begin{aligned}\alpha(u) &= \prod_{k=1}^{L} f_{\bar{\mathcal{D}}_{c_{i_k}}}(u - v_{i_k}) f_{\mathcal{D}_{c_{j_k}}}(u - v_{j_k}) \frac{u - v_{j_k} + c_{j_k}}{u - v_{j_k}} = \prod_{k=1}^{L} \frac{u - v_{j_k} + c_{j_k}}{u - v_{j_k}}\,, \\ \delta(u) &= \prod_{k=1}^{L} f_{\bar{\mathcal{D}}_{c_{i_k}}}(u - v_{i_k}) f_{\mathcal{D}_{c_{j_k}}}(u - v_{j_k}) \frac{u - v_{i_k} + c_{i_k}}{u - v_{i_k}} = \prod_{k=1}^{L} \frac{u - v_{j_k} + 1}{u - v_{j_k} + 1 + c_{j_k}}\,.\end{aligned} \tag{3.106}$$

To show the last equality for each eigenvalue, we observe that the conditions (3.105) for the k-th line of the Baxter lattice are equivalent to those for the two-site invariant in (2.59). Therefore the normalization factors in the eigenvalues trivialize like in case of the two-site invariant in (2.60). Obviously, (3.106) solves the functional relation (3.36). The other functional relation (3.37) has the unique solution

$$Q(u) = \prod_{k=1}^{L} \frac{\Gamma(u - v_{j_k} + c_{j_k} + 1)}{\Gamma(u - v_{j_k} + 1)} = \prod_{k=1}^{L} \prod_{l=1}^{c_{j_k}} (u - v_{j_k} + l) \tag{3.107}$$

because we demand the Q-function to be of the form (3.22). The zeros of this function are the Bethe roots

$$\begin{aligned} u_l &= v_{j_1} - l \quad \text{for} \quad l = 1, \ldots, c_{j_1}\,, \\ u_{l + c_{j_1}} &= v_{j_2} - l \quad \text{for} \quad l = 1, \ldots, c_{j_2}\,, \\ &\vdots \\ u_{l + c_{j_{L-1}}} &= v_{j_L} - l \quad \text{for} \quad l = 1, \ldots, c_{j_L}\,. \end{aligned} \tag{3.108}$$

These roots form L strings in the complex plane. The k-th line of the Baxter lattice yields a string of $c_{j_k} = -c_{i_k} \in \mathbb{N}$ uniformly spaced Bethe roots lying between the inhomogeneities v_{i_k} and v_{j_k}. Thus the representation $\mathcal{T}_k = \bar{\mathcal{D}}_{c_{i_k}}$ of this line determines the length of the string and the spectral parameter θ_k fixes its position in the complex plane, cf. (3.105). Next, we compute the reference state (3.17) of the associated Bethe vector using (3.105) and the form of the highest weight states in (2.41),

$$|\Omega\rangle = \prod_{k=1}^{L} (\bar{\mathbf{a}}_2^{i_k})^{c_{j_k}} (\bar{\mathbf{a}}_1^{j_k})^{c_{j_k}} |0\rangle\,. \tag{3.109}$$

Finally, the Yangian invariant $|\Psi\rangle$ is the Bethe vector (3.19). From (3.103) we know that the components of this Bethe vector are the partition functions of the Baxter lattice for different boundary conditions.

3.4.4 Relation to Perimeter Bethe Ansatz

After discussing the solution of the functional relations (3.36) and (3.37) that is associated with a Baxter lattice of L lines in the previous section 3.4.3, we show that in a special case it reproduces the perimeter Bethe ansatz of section 3.3.2. For this we first have to derive some special properties of the $\mathfrak{gl}(2)$ Lax operators. Then we restrict to a Baxter lattice where each line carries the dual of the defining representation. Finally, the algebraic Bethe vector of the associated Yangian invariant $|\Psi\rangle$ is expressed in terms of a coordinate Bethe ansatz wave function. The result is the perimeter Bethe ansatz formula (3.82) for the partition function $\mathcal{Z}(\mathbf{G}, \boldsymbol{\theta}, \mathfrak{a})$.

Let us first concentrate on the special properties of the Lax operators. These are based on a relation between the compact oscillator representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_{-c}$, which holds for $\mathfrak{gl}(2)$ but does not extend to the higher rank $\mathfrak{gl}(n)$ case. The generators (2.38) and the highest weight states (2.41) of these representations are related by

$$U\mathbf{J}_{\alpha\beta}U^{-1} = \bar{\mathbf{J}}_{\alpha\beta} + c\,\delta_{\alpha\beta}\,, \quad U|\sigma\rangle = (-1)^c|\bar{\sigma}\rangle\,, \tag{3.110}$$

with the unitary operator

$$U = e^{\frac{\pi}{2}(\bar{\mathbf{a}}_1\mathbf{a}_2 - \bar{\mathbf{a}}_2\mathbf{a}_1)} \quad \text{obeying} \quad U|0\rangle = |0\rangle\,, \quad \bar{\mathbf{a}}_1 U = U\bar{\mathbf{a}}_2\,, \quad \bar{\mathbf{a}}_2 U = -U\bar{\mathbf{a}}_1\,. \tag{3.111}$$

In what follows, it proves to be convenient to employ Lax operators with a certain fixed normalization to avoid spurious divergencies,

$$\tilde{R}_{\Box\,\mathcal{D}_c}(u-w) = (u-w)1 + \sum_{\alpha,\beta=1}^{2} E_{\alpha\beta}\bar{\mathbf{a}}_\beta\mathbf{a}_\alpha\,. \tag{3.112}$$

From these we construct a monodromy matrix with inhomogeneities w_i,

$$\tilde{M}(u) = \tilde{R}_{\Box\,\mathcal{D}_{c_1}}(u-w_1)\cdots\tilde{R}_{\Box\,\mathcal{D}_{c_N}}(u-w_N)\,. \tag{3.113}$$

The standard Lax operators (2.45) for the representation $\mathcal{D}_c$ and (2.46) for $\bar{\mathcal{D}}_{-c}$ can be reformulated in terms of (3.112) with the help of (3.110),

$$R_{\Box\,\mathcal{D}_c}(u) = \frac{f_{\mathcal{D}_c}(u)}{u}\tilde{R}_{\Box\,\mathcal{D}_c}(u)\,, \quad R_{\Box\,\bar{\mathcal{D}}_{-c}}(u) = \frac{f_{\bar{\mathcal{D}}_{-c}}(u)}{u}U\tilde{R}_{\Box\,\mathcal{D}_c}(u-c)U^{-1}\,. \tag{3.114}$$

Employing these relations, any $\mathfrak{gl}(2)$ monodromy $M(u)$ built from $R_{\Box\,\mathcal{D}_c}(u)$ and $R_{\Box\,\bar{\mathcal{D}}_{-c}}(u)$ can be expressed via $\tilde{M}(u)$, which only contains Lax operators of the type $\tilde{R}_{\Box\,\mathcal{D}_c}(u)$.

We will use this reformulation for the monodromy corresponding to a Baxter lattice with L lines defined in (3.105). To end up with the perimeter Bethe ansatz, we restrict to a lattice consisting solely of lines carrying the dual of the defining representation. Notice that according to (3.110), the defining representation of $\mathfrak{gl}(2)$ and its dual are essentially equivalent. However, choosing lines with the dual representation is more natural in our conventions. Employing the notation of (3.85) this choice means

$$\boldsymbol{\mathcal{T}} = (\bar{\Box}_1, \ldots, \bar{\Box}_L)\,. \tag{3.115}$$

We realize these representations in terms of oscillators, $\bar{\Box}_i = \bar{\mathcal{D}}_{c_{i_k}}$ with $c_{i_k} = -1$. This choice of representations together with (3.105) implies $c_{j_k} = -c_{i_k} = 1$. Consequently, the strings of Bethe roots in (3.108) reduce to single points,

$$u_k = \theta_k + 1 \quad \text{for} \quad k = 1, \ldots, L\,. \tag{3.116}$$

These Bethe roots already match those of the perimeter Bethe ansatz in (3.81). Next, the monodromy matrix defined by (3.105) and (3.115) is rewritten using (3.114) as

$$M(u)=\prod_{i=1}^{2L}\frac{1}{u-v_i}W\tilde{M}(u)W^{-1}\quad\text{with}\quad W=\prod_{k=1}^{L}U^{i_k}\,. \tag{3.117}$$

In this monodromy the normalizations of the Lax operators cancel, as we observed already after (3.106). All sites carrying a dual representation are transformed by W because the unitary operator U^{i_k} acts on site i_k. The representation labels and inhomogeneities of $\tilde{M}(u)$ in (3.113) are

$$c_i=1\,,\quad w_{i_k}=\theta_k+1\,,\quad w_{j_k}=\theta_k+2\,. \tag{3.118}$$

The inhomogeneities w_{i_k}, which stem from the dual sites of $M(u)$, are shifted by 1 with respect to the v_{i_k} in (3.105). The inhomogeneities in (3.118) match those of the perimeter Bethe ansatz in (3.81). The highest weight state $|\tilde{\Omega}\rangle$ in the total quantum space of $\tilde{M}(u)$ is derived from $|\Omega\rangle$ in (3.109) with the help of (3.110),

$$|\tilde{\Omega}\rangle=(-1)^L W^{-1}|\Omega\rangle=\bar{\mathbf{a}}_1^1\cdots\bar{\mathbf{a}}_1^N|0\rangle\,. \tag{3.119}$$

Employing (3.117) and (3.119) we can express the Bethe vector (3.19), which is constructed from the monodromy element $M_{12}(u)=B(u)$, as a Bethe vector built up from the element $\tilde{M}_{12}(u)=\tilde{B}(u)$ of the new monodromy,

$$|\Psi\rangle=(-1)^L\prod_{k=1}^{L}\prod_{i=1}^{2L}\frac{1}{u_k-v_i}W|\tilde{\Psi}\rangle\quad\text{with}\quad|\tilde{\Psi}\rangle=\tilde{B}(u_1)\cdots\tilde{B}(u_L)|\tilde{\Omega}\rangle\,. \tag{3.120}$$

Therefore, also the Yangian invariant $|\Psi\rangle$ of the Baxter lattice can be expressed in terms of $|\tilde{\Psi}\rangle$.

We continue by representing the algebraic Bethe vector $|\tilde{\Psi}\rangle$ in (3.120) using coordinate Bethe ansatz wave functions. In case of a monodromy $\tilde{M}(u)$ of the type (3.113) with representation labels $c_i=1$ at all sites, the vector reads, see e.g. [172] and appendix 3.E of [134],[6]

$$|\tilde{\Psi}\rangle=\tilde{B}(u_1)\cdots\tilde{B}(u_P)|\tilde{\Omega}\rangle=\sum_{1\leq x_1<\cdots<x_P\leq N}\Phi(\mathbf{w},\mathbf{u},\mathbf{x})\,\mathbf{J}_{21}^{x_1}\cdots\mathbf{J}_{21}^{x_P}|\tilde{\Omega}\rangle\,, \tag{3.121}$$

with $\mathfrak{gl}(2)$ generators $\mathbf{J}^i_{\alpha\beta}=\bar{\mathbf{a}}^i_\alpha\mathbf{a}^i_\beta$ and the Bethe wave function $\Phi(\mathbf{w},\mathbf{u},\mathbf{x})$ from (3.76). The arguments $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{x}$ defined in (3.75) encode respectively the inhomogeneities w_i, Bethe roots u_k and magnon positions x_k. To apply (3.121) in (3.120) for the case of Yangian invariants we need $N=2L$ sites and $P=L$ Bethe roots.

Recall the connection between the partition function and the Yangian invariant vector $|\Psi\rangle$ in (3.103),

$$\mathcal{Z}(\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\mathfrak{a})\propto\langle\mathfrak{a}|\Psi\rangle\,. \tag{3.122}$$

For the representations specified in (3.115) the possible boundary states of the Baxter lattice are $|\mathfrak{a}\rangle=|\mathfrak{a}_1\rangle\otimes\cdots\otimes|\mathfrak{a}_{2L}\rangle$ with $\mathfrak{a}_i=1,2$. Here the Gothic labels $\mathfrak{a}_i$ are identical to the Greek indices $\alpha_i=1,2$ of the oscillators which build up the states $|\mathfrak{a}_i\rangle\equiv|\alpha_i\rangle=\bar{\mathbf{a}}^i_{\alpha_i}|0\rangle$ at each site, cf. footnote 5. Inserting (3.120) and (3.121) into (3.122), the scalar product

[6] A proof of the analogous relation for more general compact $\mathfrak{gl}(2)$ representations, that are equivalent to $\mathcal{D}_c$ with $c\in\mathbb{N}$, albeit without inhomogeneities, $w_i=0$, may be found in [173].

in the latter equation reduces to that for each term in (3.121). This is non-vanishing only if the state labels $\mathfrak{a}$ satisfy the ice rule (3.74), and if $\mathbf{x}$ is given in terms of $\mathbf{G}$ and $\mathfrak{a}$ by (3.80). In this case

$$\langle\mathfrak{a}|W\mathbf{J}_{21}^{x_1}\cdots\mathbf{J}_{21}^{x_L}|\tilde{\Omega}\rangle=(-1)^{\mathcal{K}(\mathbf{G},\mathfrak{a})}\,,\tag{3.123}$$

where $\mathcal{K}(\mathbf{G},\mathfrak{a})$ is specified in (3.83) and the factor of -1 originates from sites transformed by W.

Taken together, (3.120), (3.121) and (3.123) yield the final formula for the partition function (3.122). For it to be non-zero the state labels $\mathfrak{a}$ have to obey (3.74). In this case

$$\mathcal{Z}(\mathbf{G},\boldsymbol{\mathcal{T}},\boldsymbol{\theta},\mathfrak{a})\propto\langle\mathfrak{a}|\Psi\rangle=(-1)^L\prod_{k=1}^{L}\prod_{i=1}^{2L}\frac{1}{u_k-v_i}\,(-1)^{\mathcal{K}(\mathbf{G},\mathfrak{a})}\Phi(\mathbf{w},\mathbf{u},\mathbf{x})\,,\tag{3.124}$$

where the representations in $\boldsymbol{\mathcal{T}}$ are specified by (3.115). Furthermore, the wave function arguments $\mathbf{w}$, $\mathbf{u}$, $\mathbf{x}$ are fixed in terms of the variables $\mathbf{G}$, $\boldsymbol{\theta}$, $\mathfrak{a}$ of the partition function using (3.80) and (3.81). The l.h.s. of (3.124) agrees with the perimeter Bethe ansatz expression (3.82) up to an $\mathfrak{a}$-independent normalization.

Yet it is not possible to fix this normalization factor from the Bethe ansatz. Let us now argue why the choice in (3.82) gives the correct partition function (3.73). Obviously, $\mathcal{Z}(\mathbf{G},\boldsymbol{\theta},\mathfrak{a}_0)=1$ for the particular state labels $\mathfrak{a}_0=(1,\ldots,1)$ because the Boltzmann weight in the upper left entry of the R-matrix (3.71) is equal to 1. The $\mathfrak{a}$-independent normalization chosen in (3.82) clearly reproduces this value of the partition function for $\mathfrak{a}=\mathfrak{a}_0$. From the Bethe ansatz derivation of (3.124) we already know that the $\mathfrak{a}$-dependence of (3.82) agrees with that of the partition function (3.73). This concludes the derivation of (3.82). What is more, we showed that Baxter's perimeter Bethe ansatz reviewed in section 3.3.2 is a very particular case of the Bethe ansatz for Yangian invariants that we established in section 3.2.

Chapter 4

Graßmannian Integrals and Scattering Amplitudes

After utilizing the Bethe ansatz in the preceding chapter, we develop a further method for the construction of Yangian invariants: the unitary Graßmannian integral. It is a refinement of the Graßmannian integral in the introductory section 1.3.5. In particular, we integrate over the unitary group manifold, whereas the integration contour has to be imposed "by hand" in the original proposal. Our approach is applicable for oscillator representations of the non-compact superalgebra $\mathfrak{u}(p,q|m)$. If $p=q$, we are able to change the basis from oscillators to spinor helicity-like variables. This allows us to examine the relation between the Yangian invariants obtained from our unitary Graßmannian integral and tree-level superamplitudes of $\mathcal{N}=4$ SYM.

We begin in section 4.1 by introducing a Graßmannian integral formula for oscillator representations of $\mathfrak{u}(p,q|m)$, though at first without specifying a contour. For special representation labels and inhomogeneities, and upon enforcing a unitary contour, the integral reduces to the Brezin-Gross-Witten matrix model. This observation motivates the use of the unitary contour for general values of the parameters. We prove that this contour guarantees the Yangian invariance of what we then call unitary Graßmannian integral or matrix model. We employ this method to recover several sample invariants from section 2.4. The unitary Graßmannian integral approach is complementary to the Bethe ansatz for Yangian invariants of chapter 3. On the one hand, it is limited to Yangian invariants corresponding to specific permutations in the classification of section 3.2.3, which was derived from the Bethe ansatz. On the other hand, it allows for the construction of invariants for $\mathfrak{u}(p,q|m)$, whereas the Bethe ansatz is currently limited to $\mathfrak{u}(2)$.

In section 4.2 we change the basis in the integrand of the unitary Graßmannian formula for $\mathfrak{u}(p,p|m)$ from oscillators to spinor helicity-like variables. For the bosonic oscillators this amounts to a Bargmann transformation. Such a transformation is known e.g. from the one-dimensional harmonic oscillator in quantum mechanics, where it implements the transition from Fock space to position space.

Next, in section 4.3 we apply the change of basis to the entire unitary Graßmannian integral. The resulting formula in spinor helicity-like variables is then compared to the original Graßmannian integral from section 1.3.5. Besides the presence of a unitary contour in our approach, we also work in the physical Minkowski signature, which is not the case in the original framework. In addition, our integral inherently contains deformation parameters in the form of inhomogeneities and representation labels. The branch cuts of the integrand, which caused problems with such parameters for the deformed amplitudes in section 1.3.6, disappear because of the unitary contour. To put our proposal to the test, we compare sample Yangian invariants computed with the unitary Graßmannian integral to known expressions for superamplitudes of $\mathcal{N}=4$ SYM and deformations thereof.

Lastly, some additional material on the unitary Graßmannian integral is deferred to appendix B.

4.1 Graßmannian Integral in Oscillator Variables

4.1.1 Graßmannian Formula

We delve into this chapter by directly presenting one of the main results, a Graßmannian integral formula for Yangian invariants with oscillator representations of the non-compact superalgebra $\mathfrak{u}(p,q|r+s)$. Recall the notation $p+q=n$ and $r+s=m$ from section 2.3. We motivate our formula by combining our knowledge of the Graßmannian integral for deformed scattering amplitudes (1.54) with that of the simple two-site oscillator sample invariant (2.94). Here we merely state the resulting formula. Its implications, some refinements and examples will be explored in detail in the subsequent sections. In particular, the proof of its Yangian invariance is deferred to section 4.1.2.

We consider the monodromy $M_{N,K}(u)$ in (2.89) with $N=2K$ sites, out of which the first K carry a "dual" oscillator representation $\bar{\mathcal{D}}_{c_i}$ and the remaining $K=N-K$ sites carry an "ordinary" one $\mathcal{D}_{c_i}$. The normalization factors of the Lax operators (2.18) are chosen to be trivial, i.e. $f_{\mathcal{D}_{c_i}} = f_{\bar{\mathcal{D}}_{c_i}} = 1$. A Yangian invariant for this monodromy is given by the *Graßmannian integral formula*

$$|\Psi_{N,K}\rangle = \int \mathrm{d}\mathcal{C} \frac{e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle}{(\det \mathcal{C})^{q-s}(1,\ldots,K)^{1+v_K^+ - v_1^-}\cdots(N,\ldots,K-1)^{1+v_{K-1}^+ - v_N^-}} \,. \tag{4.1}$$

Here the numerator can be understood as a matrix generalization of that of the two-site sample invariant (2.94). The single contractions of oscillators in the exponent in that formula are replaced by the $K\times K$ matrices

$$\mathbf{I}_{\substack{\bullet\\\circ}} = \begin{pmatrix} (1 \overset{\bullet}{\circ} K+1) & \cdots & (1 \overset{\bullet}{\circ} N) \\ \vdots & & \vdots \\ (K \overset{\bullet}{\circ} K+1) & \cdots & (K \overset{\bullet}{\circ} N) \end{pmatrix} . \tag{4.2}$$

These matrices contain all possible contractions of oscillators between dual and ordinary sites that we defined in (2.90),

$$(k \bullet l) = \sum_{\mathsf{A}} \bar{\mathbf{A}}^l_{\mathsf{A}} \bar{\mathbf{A}}^k_{\mathsf{A}} \,, \quad (k \circ l) = \sum_{\dot{\mathsf{A}}} \bar{\mathbf{A}}^l_{\dot{\mathsf{A}}} \bar{\mathbf{A}}^k_{\dot{\mathsf{A}}} \,. \tag{4.3}$$

These entries of the matrices $\mathbf{I}_\bullet$ and $\mathbf{I}_\circ$ are respectively $\mathfrak{u}(p|r)$ and $\mathfrak{u}(q|s)$ invariant, cf. (2.91). We may think of these invariants of compact subalgebras of $\mathfrak{u}(p,q|r+s)$ as "elementary building blocks" of the Yangian invariant. The denominator of (4.1) is analogous to that of the Graßmannian integral for deformed scattering amplitudes (1.54). It contains the minors $(i,\ldots,i+K-1)$ of the $K\times N$ matrix C defined in (1.41). However, notice the extra factor $(\det\mathcal{C})^{q-s} = (N-K+1,\ldots,N)^{q-s}$, which depends on the symmetry algebra. The gauge fixing of the matrix C in (1.41) corresponds to the order of dual and ordinary sites. Furthermore, the integral in (4.1) is over the holomorphic K^2-form $\mathrm{d}\mathcal{C} = \bigwedge_{k,l}\mathrm{d}C_{kl}$, which we already encountered in (1.54). The $2N$ parameters v_i^+, v_i^- appearing as exponents of the minors are related to the $2N$ parameters v_i, c_i of the monodromy (2.89) by, cf. [108],

$$v_i^{\pm} = v_i' \pm \frac{c_i}{2}\,, \quad v_i' = v_i - \frac{c_i}{2} + \begin{cases} n-m-1 & \text{for} \quad i = 1, \dots, K\,, \\ 0 & \text{for} \quad i = K+1, \dots, N\,. \end{cases} \tag{4.4}$$

Finally, for $|\Psi_{N,K}\rangle$ in (4.1) to be Yangian invariant, the parameters v_i^+, v_i^- have to satisfy the N relations

$$v_{i+K}^+ = v_i^- \tag{4.5}$$

for $i = 1, \dots, N$. These relations are analogous to (1.56) for the deformed amplitudes.[1] Recall that while the inhomogeneities v_i are complex numbers, the labels c_i of the oscillator representations $\mathcal{D}_{c_i}$ and $\bar{\mathcal{D}}_{c_i}$ have to be integers, see section 2.3. This yields further constraints on the parameters v_i^+, v_i^-. For now, this completes the specification of the Graßmannian integral formula (4.1).

One obvious omission in this specification is the choice of a multi-dimensional contour of integration in (4.1). The proof of Yangian invariance in the following section only assumes that certain boundary terms vanish upon integration by parts, which is satisfied in particular for closed contours. The choice of a suitable integration contour will be of paramount importance in the sections thereafter.

We add some further remarks. The condition $N = 2K$ guarantees $\mathcal{C}$ to be a square matrix. Thus it is sensible to use its inverse in (4.1). In the compact special case $\mathfrak{u}(p, 0|r)$ we have $\mathbf{I}_\circ = 0$, thus $\mathcal{C}^{-1}$ is absent from (4.1) and the Graßmannian integral yields Yangian invariants also for $N \neq 2K$. However, we do not elaborate on the compact case in this work. We note that because of $\mathbf{I}_\circ = 0$, the *compact* case of (4.1) is reminiscent of the link representation of scattering amplitudes, cf. [92]. It is different though, as the amplitudes transform under the *non-compact* superconformal algebra. Let us also point out a relation between the Graßmannian integral (4.1) and the Bethe ansatz for Yangian invariants of section 3.2. In the $\mathfrak{u}(2, 0|0)$ case the redefinition of variables in (4.4) reduces to (3.65). This equation was important to obtain the classification (3.68) of the solutions to the functional relation (3.36) in terms of permutations. We can identify (4.5) with (3.68) for the special permutation $\sigma(i) = i + K$. Thus we expect that for $\mathfrak{u}(2)$ the Yangian invariants produced by the Graßmannian integral (4.1) are precisely those which we already know from the Bethe ansatz. We will verify this explicitly for some examples in section 4.1.5.

4.1.2 Proof of Yangian Invariance

Here we prove the Yangian invariance of the Graßmannian integral (4.1) for the invariant $|\Psi_{N,K}\rangle$ with $N = 2K$ sites and oscillator representations of the non-compact superalgebra $\mathfrak{u}(p, q|r + s)$. We will verify the expanded form (2.27) of the Yangian invariance condition. As argued there, it is sufficient to check this equation for the expansion coefficients $M_{\mathcal{AB}}^{(1)}$ and $M_{\mathcal{AB}}^{(2)}$ of the monodromy elements $M_{\mathcal{AB}}(u)$. With straightforward modifications the following proof also applies to the compact case with $q = s = 0$ where $\mathbf{I}_\circ = 0$ and $N \neq 2K$ is possible.

Let us start with the ansatz

$$|\Phi\rangle = e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle\,, \tag{4.6}$$

which we recognize as the exponential function in (4.1). We want to show that this ansatz satisfies (2.27) with the Yangian generators $M_{\mathcal{AB}}^{(1)}$, that is to say $\mathfrak{gl}(n|m)$ invariance. Using

[1] Because $N = 2K$, the relations (4.5) are equivalent to $v_{i+K}^- = v_i^+$, which is exactly the form of (1.56).

the expression (2.20) of these Yangian generators in terms of $\mathfrak{gl}(n|m)$ generators for the monodromy (2.89) yields

$$M^{(1)}_{\mathcal{AB}} = \sum_{k=1}^{K} \bar{\mathbf{J}}^k_{\mathcal{BA}} + \sum_{l=K+1}^{N} \mathbf{J}^l_{\mathcal{BA}} \,. \tag{4.7}$$

The generators $\bar{\mathbf{J}}^k_{\mathcal{BA}}$ of the "dual" oscillator representation $\bar{\mathcal{D}}_{c_k}$ are given in (2.36). Likewise, the $\mathbf{J}^l_{\mathcal{BA}}$ defined in (2.34) generate the "ordinary" representation $\mathcal{D}_{c_l}$. To evaluate the action of the operator (4.7) on the ansatz (4.6) we compute

$$\begin{aligned}
(\bar{\mathbf{J}}^k_{\mathcal{AB}})\,|\Phi\rangle &= \left(\begin{array}{c:c} -\sum_w \bar{\mathbf{A}}^w_{\mathsf{A}}\,\bar{\mathbf{A}}^k_{\mathsf{B}}\,C_{kw} & -\sum_{w,w'} \bar{\mathbf{A}}^w_{\mathsf{A}}\,\bar{\mathbf{A}}^{w'}_{\dot{\mathsf{B}}}\,D_{w'k}C_{kw} \\ \hdashline (-1)^{|\dot{\mathsf{A}}|}\bar{\mathbf{A}}^k_{\dot{\mathsf{A}}}\,\bar{\mathbf{A}}^k_{\mathsf{B}} & (-1)^{|\dot{\mathsf{A}}|}(\sum_w \bar{\mathbf{A}}^k_{\dot{\mathsf{A}}}\,\bar{\mathbf{A}}^w_{\dot{\mathsf{B}}}\,D_{wk} + \delta_{\dot{\mathsf{A}}\dot{\mathsf{B}}}) \end{array}\right)|\Phi\rangle\,,\\
(\mathbf{J}^l_{\mathcal{AB}})\,|\Phi\rangle &= \left(\begin{array}{c:c} \sum_v \bar{\mathbf{A}}^l_{\mathsf{A}}\,\bar{\mathbf{A}}^v_{\mathsf{B}}\,C_{vl} & \bar{\mathbf{A}}^l_{\mathsf{A}}\,\bar{\mathbf{A}}^l_{\dot{\mathsf{B}}} \\ \hdashline -(-1)^{|\dot{\mathsf{A}}|}\sum_{v,v'} \bar{\mathbf{A}}^v_{\dot{\mathsf{A}}}\,\bar{\mathbf{A}}^{v'}_{\mathsf{B}}C_{v'l}\,D_{lv} & -(-1)^{|\dot{\mathsf{A}}|}(\sum_v \bar{\mathbf{A}}^v_{\dot{\mathsf{A}}}\,\bar{\mathbf{A}}^l_{\dot{\mathsf{B}}}\,D_{lv} + \delta_{\dot{\mathsf{A}}\dot{\mathsf{B}}}) \end{array}\right)|\Phi\rangle\,,
\end{aligned} \tag{4.8}$$

where components of the matrix $\mathcal{C}^{-1}$ are denoted by D_{lk}. Here and in the remainder of this proof the indices k, v, v' always take the values $1, \ldots, K$ while l, w, w' are in the range $K+1, \ldots, N$. Now one immediately obtains

$$M^{(1)}_{\mathcal{AB}}\,|\Phi\rangle = 0\,. \tag{4.9}$$

Hence (2.27) with the Yangian generators $M^{(1)}_{\mathcal{AB}}$ holds for the ansatz (4.6).

However, each site of the ansatz (4.6) does not yet transform in an irreducible representation of the superalgebra $\mathfrak{u}(p,q|r+s)$. In fact, (4.6) is not an eigenstate of the central elements $\mathbf{C}^l = \sum_{\mathcal{A}=1}^{n+m} \mathbf{J}^l_{\mathcal{AA}}$ and $\bar{\mathbf{C}}^k = \sum_{\mathcal{A}=1}^{n+m} \bar{\mathbf{J}}^k_{\mathcal{AA}}$ that were defined in (2.35) and (2.37), respectively. To obtain eigenstates we have to pick special linear combinations of the ansatz (4.6),

$$|\Psi_{N,K}\rangle = \int \mathrm{d}\mathcal{C}\,\mathscr{F}(\mathcal{C})\,|\Phi\rangle\,. \tag{4.10}$$

It turns out to be suitable to choose an integrand that contains only consecutive minors of the matrix C defined in (1.41),

$$\mathscr{F}(\mathcal{C}) = \frac{1}{(1,\ldots,K)^{1+\alpha_1}\cdots(N,\ldots,K-1)^{1+\alpha_N}} \tag{4.11}$$

with arbitrary complex constants α_i. With this integrand the ansatz (4.10) is an eigenstate of the central elements,

$$\bar{\mathbf{C}}^k\,|\Psi_{N,K}\rangle = \left(q - s - \sum_{i=k+1}^{k+N-K}\alpha_i\right)|\Psi_{N,K}\rangle\,, \quad \mathbf{C}^l\,|\Psi_{N,K}\rangle = \left(-q + s + \sum_{i=l-K+1}^{l}\alpha_i\right)|\Psi_{N,K}\rangle\,. \tag{4.12}$$

To show this property we assumed that upon integration by parts the boundary terms vanish. Furthermore, we employed the identity

$$\frac{\mathrm{d}}{\mathrm{d}C_{kl}}e^{\mathrm{tr}(\mathcal{C}\mathbf{I}^t_\bullet + \mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle = \Big((k\bullet l) - \sum_{v,w} D_{wk}D_{lv}(v\circ w)\Big)e^{\mathrm{tr}(\mathcal{C}\mathbf{I}^t_\bullet + \mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle\,, \tag{4.13}$$

which is easily verified taking into account $\frac{\mathrm{d}}{\mathrm{d}C_{kl}}D_{wv} = -D_{wk}D_{lv}$. In addition, in evaluating derivatives of the minors in $\mathscr{F}(\mathcal{C})$ we used, cf. [94, 95],

$$\sum_w C_{kw} \frac{\mathrm{d}}{\mathrm{d}C_{kw}} (i, \ldots, i+K-1)^{1+\alpha_i} = (1+\alpha_i)\,(i, \ldots, i+K-1)^{1+\alpha_i} \tag{4.14}$$

for $i = k+1, \ldots, k+N-K$. For other values of i the left hand side in (4.14) vanishes due to the gauge fixing of C in (1.41).

Next, we turn our attention to the invariance condition (2.27) with the Yangian generators $M^{(2)}_{\mathcal{AB}}$. From the commutation relations (2.16) with $r = 2$ and $s = 1$ one sees that if a state $|\Psi\rangle$ is annihilated by all $M^{(1)}_{\mathcal{AB}}$ and by one of the generators $M^{(2)}_{\mathcal{AB}}$, e.g. by $M^{(2)}_{11}$, then it is annihilated by all $M^{(2)}_{\mathcal{AB}}$. Thus in our case it is sufficient to verify (2.27) for one of the four blocks of generators, say for $M^{(2)}_{\mathsf{AB}}$. Expressions for these generators can be found in (2.20). We compute the action of all terms appearing therein on our ansatz (4.6),

$$\begin{aligned}
\sum_{\mathcal{I}} (-1)^{|\mathcal{I}|} \mathbf{J}^l_{\mathsf{B}\mathcal{I}} \bar{\mathbf{J}}^k_{\mathcal{I}\mathsf{A}} |\Phi\rangle &= -\bar{\mathbf{A}}^l_{\mathsf{B}}\, \bar{\mathbf{A}}^k_{\mathsf{A}} \Big(\sum_{v,w} C_{vl} C_{kw} \frac{\mathrm{d}}{\mathrm{d}C_{vw}} + (p-r)\, C_{kl} \Big) |\Phi\rangle\,, \\
\sum_{\mathcal{I}} (-1)^{|\mathcal{I}|} \bar{\mathbf{J}}^{k'}_{\mathsf{B}\mathcal{I}} \bar{\mathbf{J}}^k_{\mathcal{I}\mathsf{A}} |\Phi\rangle &= \sum_{w,w'} \bar{\mathbf{A}}^w_{\mathsf{B}}\, \bar{\mathbf{A}}^k_{\mathsf{A}}\, C_{k'w}\, C_{kw'} \frac{\mathrm{d}}{\mathrm{d}C_{k'w'}} |\Phi\rangle\,, \\
\sum_{\mathcal{I}} (-1)^{|\mathcal{I}|} \mathbf{J}^{l'}_{\mathsf{B}\mathcal{I}} \mathbf{J}^l_{\mathcal{I}\mathsf{A}} |\Phi\rangle &= \sum_{v,v'} \bar{\mathbf{A}}^{l'}_{\mathsf{B}}\, \bar{\mathbf{A}}^v_{\mathsf{A}}\, C_{vl}\, C_{v'l'} \frac{\mathrm{d}}{\mathrm{d}C_{v'l}} |\Phi\rangle
\end{aligned} \tag{4.15}$$

for $k \neq k'$ and $l \neq l'$, and furthermore

$$\Big(\sum_k v_k\, \bar{\mathbf{J}}^k_{\mathsf{BA}} + \sum_l v_l\, \mathbf{J}^l_{\mathsf{BA}} \Big) |\Phi\rangle = \sum_{k,l} \bar{\mathbf{A}}^l_{\mathsf{B}}\, \bar{\mathbf{A}}^k_{\mathsf{A}}\, C_{kl}\, (v_l - v_k)\, |\Phi\rangle\,. \tag{4.16}$$

Making use of these formulas we can evaluate the action on (4.10),

$$M^{(2)}_{\mathcal{AB}} |\Psi_{N,K}\rangle = \sum_{k,l} \Big(v_l - v_k - p + r + 1 - \sum_{i=l-K+1}^{k+N-K} \alpha_i \Big)\, \bar{\mathbf{A}}^l_{\mathsf{B}}\, \bar{\mathbf{A}}^k_{\mathsf{A}} \int \mathrm{d}\mathcal{C} \mathscr{F}(\mathcal{C})\, C_{kl}\, |\Phi\rangle\,. \tag{4.17}$$

Here we assumed once more that the boundary terms of the integration by parts vanish. Furthermore, we used (4.13) and properties of the minors in $\mathscr{F}(\mathcal{C})$ similar to (4.14). To ensure Yangian invariance of the ansatz, the parameters α_i have to be chosen such that the bracket in (4.17) vanishes.

In conclusion, for the ansatz (4.10) to be Yangian invariant, the parameters v_i, c_i of the monodromy and the α_i appearing in this ansatz have to obey the equations obtained from (4.12) and (4.17),

$$c_k = q - s - \sum_{i=k+1}^{k+N-K} \alpha_i\,, \quad c_l = -q + s + \sum_{i=l-K+1}^{l} \alpha_i\,, \quad v_k - v_l = -p + r + 1 - \sum_{i=l-K+1}^{k+N-K} \alpha_i \tag{4.18}$$

for $k = 1, \ldots K$ and $l = K+1, \ldots, N$. These equations are conveniently addressed after changing from the variables v_i, c_i to v_i^+, v_i^- with (4.4). In these variables they are solved by

$$\alpha_i = v^+_{i+K-1} - v^-_i + (q-s)\, \delta_{i,N-K+1} \tag{4.19}$$

and imposing the N constraints in (4.5). Equation (4.19) turns the ansatz (4.10) into the Graßmannian integral formula (4.1). This concludes the proof of its Yangian invariance.

4.1.3 Unitary Matrix Models

In the introductory section 1.3 we saw that the Graßmannian integral for $\mathcal{N}=4$ SYM scattering amplitudes (1.43) is that special case of the deformed integral (1.54) where the exponents of all minors are equal to 1. Here we investigate a special case of our Graßmannian integral formula (4.1) in oscillator variables. We choose deformations parameters $v_i^{\pm}$ such that the exponents of almost all of the minors are identical to 0. Although this choice seems trivial from an amplitudes perspective, it reveals an interesting connection between the Graßmannian integral (4.1) and certain unitary matrix models. The *integrand* of (4.1) reduces to that of the Brezin-Gross-Witten matrix model or even a slight generalization thereof, the Leutwyler-Smilga model. To identify the entire *integral* in (4.1) with these unitary matrix models, we choose the contour of integration to be the unitary group manifold. This "unitary contour" will be of pivotal importance in the rest of this chapter. Furthermore, in the special case explored here the Graßmannian integral (4.1) can be computed easily by applying well established matrix model techniques. In this way, we obtain a representation of these Yangian invariants in terms of Bessel functions.

In order to reduce (4.1) with $N=2K$ to the Leutwyler-Smilga integral, we restrict to a special solution of the constraints in (4.5) on the deformation parameters $v_i^{\pm}$. The solution has to be such that all minors in (4.1), except for $(1,\dots,K)=1$ and $(N-K+1,\dots,N)=\det\mathcal{C}$, have a vanishing exponent. A short calculation shows that this solution depends only on two parameters $v\in\mathbb{C}, c\in\mathbb{Z}$. It is given by

$$\begin{aligned} v_i &= v-c-n+m+1+(i-1)\,, & c_i &= -c & &\text{for} & i&=1,\dots,K\,,\\ v_i &= v+(i-K-1)\,, & c_i &= c & &\text{for} & i&=K+1,\dots,2K\,. \end{aligned} \tag{4.20}$$

Here we used (4.4) to change from the variables v_i^+, v_i^- employed in (4.1) to the variables v_i, c_i. Let us now focus on the measure $\mathrm{d}\mathcal{C}=\bigwedge_{k,l}\mathrm{d}\mathcal{C}_{k,l}$ in (4.1). One readily verifies that

$$[\mathrm{d}\mathcal{C}] = \chi_K\,\frac{\mathrm{d}\mathcal{C}}{(\det\mathcal{C})^K}\,, \tag{4.21}$$

with a constant number $\chi_K\in\mathbb{C}$, is invariant under $\mathcal{C}\mapsto\mathcal{V}\mathcal{C}$ and $\mathcal{C}\mapsto\mathcal{C}\mathcal{V}$ for any constant matrix $\mathcal{V}\in GL(\mathbb{C}^K)$. Because of these properties, for unitary $\mathcal{C}$ the differential form $[\mathrm{d}\mathcal{C}]$ defined in (4.21) gives rise to the Haar measure on the unitary group $U(K)$, cf. [174, 175]. The normalization χ_K is chosen such that $\int_{U(K)}[\mathrm{d}\mathcal{C}]=1$. We select a "unitary contour" in the Graßmannian integral (4.1) by demanding $\mathcal{C}^\dagger=\mathcal{C}^{-1}$. This allows us to express the Yangian invariant with the special choice of deformation parameters (4.20) as

$$|\Psi_{2K,K}\rangle = \chi_K^{-1}\int_{U(K)}[\mathrm{d}\mathcal{C}]\frac{e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^\dagger)}|0\rangle}{(\det\mathcal{C})^{c+q-s}}\,, \tag{4.22}$$

where $c\in\mathbb{Z}$ is a free parameter. Equation (4.22) is known as *Leutwyler-Smilga model* [176], where the matrices $\mathbf{I}_\bullet^t$ and $\mathbf{I}_\circ$ are considered as sources. For $c+q-s=0$ it becomes the *Brezin-Gross-Witten model* [177, 178, 179]. Remarkably, the integral (4.22) can be computed exactly. For two *independent* source matrices $\mathbf{I}_\bullet^t$ and $\mathbf{I}_\circ$ this was achieved in [180] using the character expansion methods of [181],

$$|\Psi_{2K,K}\rangle = \chi_K^{-1}\prod_{j=0}^{K-1} j!\,\frac{(\det\mathbf{I}_\bullet^t)^{c+q-s}}{\Delta(\mathbf{I}_\circ\mathbf{I}_\bullet^t)}\det\left(\frac{I_{k+c+q-s-K}\big(2\sqrt{(\mathbf{I}_\circ\mathbf{I}_\bullet^t)_l}\big)}{\sqrt{(\mathbf{I}_\circ\mathbf{I}_\bullet^t)_l}^{\,k+c+q-s-K}}\right)_{k,l}|0\rangle\,. \tag{4.23}$$

The entries of the matrices $\mathbf{I}_\bullet^t$ and $\mathbf{I}_\circ$ are bosonic even in the supersymmetric setting, see the discussion after (2.90). Assuming the matrix $\mathbf{I}_\circ\mathbf{I}_\bullet^t$ to be diagonalizable, we denote its

l-th eigenvalue by $(\mathbf{I}_\circ \mathbf{I}_\bullet^t)_l$. Furthermore, $\Delta(\mathbf{I}_\circ \mathbf{I}_\bullet^t) = \det(({\mathbf{I}_\circ \mathbf{I}_\bullet^t)_l}^{k-1})_{k,l}$ is the Vandermonde determinant. The formula (4.23) involving a determinant of Bessel functions $I_\nu(x)$ generalizes the single Bessel function that we found for the sample Yangian invariant $|\Psi_{2,1}\rangle$ in (2.92).

We conclude this section by adding some background material on the unitary matrix integral (4.22). It is of relevance in multiple physical contexts. The Brezin-Gross-Witten model appears in two-dimensional massless lattice QCD, see [177]. The partition function of this gauge theory can be reduced to (4.22) with a vanishing exponent of $\det \mathcal{C}$. Both source matrices $\mathbf{I}_\bullet^t = \mathbf{I}_\circ \propto 1_K$ are multiples of the theory's coupling constant and the description is valid at any value of this coupling. The unitary group $U(K)$ corresponds to the gauge group of the Yang-Mills field. Let us move on to a different context in which the integral (4.22) occurs. The Leutwyler-Smilga model describes four-dimensional continuum QCD with non-vanishing quark masses in a certain low energy regime. In this theory the partition function in a sector with a fixed topological charge of the gauge field is given by (4.22), where the exponent of $\det \mathcal{C}$ corresponds to that charge. Furthermore, the group $U(K)$ is associated with the flavor symmetry of the K quarks. The source matrices $\mathbf{I}_\bullet^t$ and $\mathbf{I}_\circ$ are parametrized by the quark masses. A more detailed account on this interpretation of the integral (4.22) is provided in the lecture notes [182]. Our interest in this integral is mostly of mathematical nature. Basics of the group theoretical character expansion method, which yields the determinant formula (4.23), are discussed e.g. in the concise review [183].

Numerous matrix models are long known to be related to classically integrable hierarchies of partial differential equations, see the reviews [10, 11] and references therein. This notion of integrability is closely linked to the Korteweg-de Vries equation, which we encountered as a $1+1$-dimensional model for waves in shallow water right at the beginning of this thesis in section 1.1. There exists an integrable generalization of this model to $2+1$ dimensions, the Kadomtsev-Petviashvili (KP) equation, see e.g. [9, 184]. Besides its interpretation as a model for water waves, it appears in multiple further physical contexts. In addition, it is of importance because many other integrable differential equations can be obtained from this equation by means of a symmetry reduction. Notably, the KP equation was found to belong to an infinite set of compatible integrable partial differential equations, the KP hierarchy, see the substantial review in [185]. Solutions of this hierarchy are given in the form of so-called τ-functions. After this digression, we return to the matrix models encountered in this section. The partition function of the Brezin-Gross-Witten model is known to be a τ-function of the KP hierarchy [186]. This also applies to the partition function of the slightly more general Leutwyler-Smilga model, cf. [187]. The determinant representation (4.23) of the partition function is the key to establish these relations. The connection between this integrable structure and the Yangian invariance of these models, that is investigated in this thesis, seems to be far from obvious. It would be interesting to clarify this connection.

4.1.4 Unitary Graßmannian Matrix Models

We just established that the choice of a "unitary contour" together with *special* deformation parameters $v_i^\pm$ reduces the Graßmannian integral (4.1) to a well-known unitary matrix model. In what follows, we show that this contour remains appropriate for *general* deformation parameters. This leads to a, to the best of our knowledge, novel class of unitary Graßmannian matrix models.

4.1.4.1 Single-Valuedness of Integrand

The multi-dimensional contour for the Graßmannian integral (4.1) should be closed. This ensures that the boundary terms in the proof of its Yangian invariance in section 4.1.2 vanish. Choosing the contour to be the unitary group manifold $U(K)$ seems to assure this because it is compact. However, we also have to verify that the integrand of (4.1) is a single-valued function on this contour. Otherwise, the compactness of the contour does not imply that the boundary terms vanish. Due to the complex exponents $v_i^\pm$ of the minors and the resulting branch cuts, the single-valuedness of the integrand in (4.1) is far from obvious. In fact, we have to modify the integrand in a minute way to be able to prove that it is single-valued.

We transcribe the Graßmannian integral for $N = 2K$ and the symmetry algebra $\mathfrak{u}(p,q|r+s)$ from (4.1),

$$|\Psi_{2K,K}\rangle = \chi_K^{-1} \int_{U(K)} [\mathrm{d}\mathcal{C}]\, \mathscr{F}(\mathcal{C})\, e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^\dagger)} |0\rangle\,. \tag{4.24}$$

Here we imposed the unitary contour $\mathcal{C}^{-1} = \mathcal{C}^\dagger$. In addition, we expressed $\mathrm{d}\mathcal{C}$ in terms of the Haar measure $[\mathrm{d}\mathcal{C}]$ via (4.21). Most importantly, the integrand containing the minors of the Graßmannian matrix C reads

$$\begin{aligned}
\mathscr{F}(\mathcal{C})^{-1} = {} & (\det \mathcal{C})^{q-s-K} \\
& \cdot (1,\ldots,K)^{1+v_K^+ - v_1^-} \qquad\qquad \cdots \quad (K,\ldots,2K-1)^{1+v_{2K-1}^+ - v_K^-} \\
& \cdot \left(\frac{(K+1,\ldots,2K)}{\det\mathcal{C}}\right)^{1+v_{2K}^+ - v_{K+1}^-} \cdots \quad \left(\frac{(2K,\ldots,K-1)}{\det\mathcal{C}}\right)^{1+v_{K-1}^+ - v_{2K}^-} \\
& \cdot (\det\mathcal{C})^{1+v_{2K}^+ - v_{K+1}^-} \qquad\qquad \cdots \quad (\det\mathcal{C})^{1+v_{K-1}^+ - v_{2K}^-}\,.
\end{aligned} \tag{4.25}$$

In this formula the power function is defined using the principal value of the complex logarithm, i.e. $z^v = e^{v \operatorname{Log} z} = e^{v(\ln|z| + i \operatorname{Arg} z)}$ with $\operatorname{Arg} z \in (-\pi, \pi]$. Note that for integer deformation parameters $v_i^\pm$ the integrand in (4.25) reduces to the original one in (4.1) because the factors of $\det\mathcal{C}$ in the third and fourth line cancel. However, for generic complex parameters the analytic structure of these two integrands differs.

The single-valuedness is still not obvious in (4.25). We will show it now and end up with a form of the integrand where it is manifest. We start by expressing the minors of the $K \times 2K$ matrix C defined in (1.41) in terms of those of the $K \times K$ matrix $\mathcal{C}$,

$$(i,\ldots,i+K-1) = (-1)^{(K-i+1)(i-1)} \begin{cases} [1,\ldots,i-1] & \text{for} \quad i = 1,\ldots,K\,, \\ [i-K,\ldots,K] & \text{for} \quad i = K+1,\ldots,2K\,. \end{cases} \tag{4.26}$$

In this formula the principal minor of $\mathcal{C}$ corresponding to the rows and columns i to j is denoted $[i,\ldots,j]$, e.g. $[\;] = 1$, $[1] = \mathcal{C}_{1\,K+1}$ and $[1,\ldots,K] = \det\mathcal{C}$. Furthermore, using the unitarity of $\mathcal{C}$ we obtain

$$[i+1,\ldots,K] = \overline{[1,\ldots,i]}\,\det\mathcal{C}\,, \tag{4.27}$$

where the bar denotes complex conjugation, see e.g. [188] and the reference mentioned therein. This identity can be proven using a block decomposition of $\mathcal{C}$. It turns out to be of great utility here and in the remainder of the chapter. We will use it frequently without explicit reference. Next, using (4.4) and (4.5) the exponents of the minors in (4.25) are expressed in terms of the variables $v_i \in \mathbb{C}$ and $c_i \in \mathbb{Z}$,

$$
\begin{aligned}
1 + v_K^+ - v_1^- &= 1 + v_K - v_1 + c_1 \,, \\
1 + v_{K+1}^+ - v_2^- &= 1 + v_1 - v_2 - c_1 + c_2 \,, \\
&\vdots \\
1 + v_{2K-1}^+ - v_K^- &= 1 + v_{K-1} - v_K - c_{K-1} + c_K \,, \\
1 + v_{2K}^+ - v_{K+1}^- &= 1 + v_K - v_1 - c_K \,, \\
1 + v_1^+ - v_{K+2}^- &= 1 + v_1 - v_2 \,, \\
&\vdots \\
1 + v_{K-1}^+ - v_{2K}^- &= 1 + v_{K-1} - v_K \,.
\end{aligned}
\tag{4.28}
$$

These variables obey

$$
\begin{aligned}
&v_{K+1} = v_1 + n - m - 1 - c_1 \,, \qquad \ldots, \quad v_{2K} = v_K + n - m - 1 - c_K \,, \\
&c_{K+1} = -c_1 \,, \qquad\qquad\qquad\qquad\; \ldots, \quad c_{2K} = -c_K \,.
\end{aligned}
\tag{4.29}
$$

These steps allow us to write the integrand (4.25) as

$$
\begin{aligned}
\mathscr{F}(\mathcal{C})^{-1} =& (-1)^{(c_1+\cdots+c_K)(K+1)} (\det \mathcal{C})^{q-s-c_K} \\
&\cdot |[1]|^{2(1+v_1-v_2)} [1]^{c_2-c_1} \cdots |[1,\ldots,K-1]|^{2(1+v_{K-1}-v_K)} [1,\ldots,K-1]^{c_K-c_{K-1}} \,.
\end{aligned}
\tag{4.30}
$$

This function of $\mathcal{C} = (C_{kl})$ is manifestly single-valued as only non-negative numbers are exponentiated to non-integer powers. Together with the formal proof of section 4.1.2 this shows the Yangian invariance of (4.24) with the integrand (4.25) or equivalently with (4.30). The integral (4.24) with (4.30) is a novel matrix model that we refer to as *unitary Graßmannian matrix model*. It generalizes the well established Brezin-Gross-Witten and Leutwyler-Smilga model in (4.22) by including principal minors of the unitary matrix $\mathcal{C}$ other than $\det \mathcal{C}$.

We append some remarks. Note the crucial importance of *integer* representation labels c_i in the derivation of (4.30). Hence we stay within the class of oscillator representations introduced in section 2.3. This is in contrast to previous attempts [110] to find a suitable contour for the Graßmannian integral (1.54) for deformed $\mathcal{N} = 4$ SYM scattering amplitudes, where the representation labels are typically *complex* numbers. Let us also remark that the convergence of the integral (4.24) is not guaranteed because some minors might vanish. Furthermore, a function of the form of $\mathscr{F}(\mathcal{C})$ in (4.30) appears in the classification of $U(K)$ representations in § 49 of [189]. We also observe that the function (4.30) is not a class function, i.e. it is not invariant under conjugation of $\mathcal{C}$ with an arbitrary unitary matrix. This is arguably the most important difference to the Leutwyler-Smilga model (4.22), where the corresponding function is just a power of $\det \mathcal{C}$. In particular, it hinders the direct application of character expansion methods [183, 181, 180] for the evaluation of (4.24).

For later use we list the explicit form of the integrand (4.30) for the simplest invariants. For $|\Psi_{2,1}\rangle$ we have

$$
\mathscr{F}(\mathcal{C})^{-1} = (\det \mathcal{C})^{q-s-c_1} \,. \tag{4.31}
$$

The integrand of $|\Psi_{4,2}\rangle$ is

$$
\mathscr{F}(\mathcal{C})^{-1} = (-1)^{c_1+c_2} (\det \mathcal{C})^{q-s-c_2} |[1]|^{2(1+v_1-v_2)} [1]^{c_2-c_1} \,. \tag{4.32}
$$

In case of the invariant $|\Psi_{6,3}\rangle$ we obtain

$$\begin{aligned}\mathscr{F}(\mathcal{C})^{-1} &= (\det \mathcal{C})^{q-s-c_3}|[1]|^{2(1+v_1-v_2)}[1]^{c_2-c_1}|[1,2]|^{2(1+v_2-v_3)}[1,2]^{c_3-c_2} \\ &= (\det \mathcal{C})^{q-s-c_2}|[1]|^{2(1+v_1-v_2)}[1]^{c_2-c_1}|\overline{[3]}|^{2(1+v_2-v_3)}\overline{[3]}^{c_3-c_2}\,.\end{aligned} \tag{4.33}$$

4.1.4.2 Parameterization of Unitary Contour

So far we analyzed the matrix integral (4.24) by making use of the unitarity of the integration variable $\mathcal{C}$. In order to eventually evaluate this integral, we resort to an explicit parameterization of $\mathcal{C}$. In this section we first briefly discuss a parameterization of the group $U(K)$ and the corresponding Haar measure. Then we work out the examples $U(1)$, $U(2)$ and $U(3)$ associated with the simplest Yangian invariants in some detail.

We formulate the unitary group as the semidirect product $U(K) = SU(K) \rtimes U(1)$, see e.g. [190]. Hence,

$$\mathcal{C} = \tilde{\mathcal{C}}\begin{pmatrix} e^{i\gamma} & 0 \\ 0 & 1_{K-1}\end{pmatrix} \tag{4.34}$$

with $\tilde{\mathcal{C}} \in SU(K)$ and $\gamma \in [0, 2\pi]$. We work with a parameterization of the $SU(K)$ factor in terms of products of $SU(2)$ matrices, which is already known since the 19th century [191], see also e.g. [192, 193, 194]. The Haar measure, cf. [174, 175], of the K^2-dimensional group $U(K)$ is obtained from the left- and right-invariant top-dimensional form

$$[\mathrm{d}\mathcal{C}] = \chi_K \frac{\mathrm{d}\mathcal{C}}{(\det \mathcal{C})^K}\,, \tag{4.35}$$

which already appeared in (4.21). In slight abuse of notation we use the symbol $[\mathrm{d}\mathcal{C}]$ for the Haar measure as well as for the form. The normalization χ_K is fixed by demanding $\int_{U(K)}[\mathrm{d}\mathcal{C}] = 1$. To evaluate (4.35) below for examples, it is helpful to state the transformation law

$$\mathrm{d}\mathcal{C} = \mathrm{d}C_{1\,K+1} \wedge \mathrm{d}C_{1\,K+2} \wedge \cdots \wedge \mathrm{d}C_{K\,2K} = \det\left(\frac{\partial \boldsymbol{C}}{\partial \boldsymbol{\phi}}\right) \mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_{K^2} \tag{4.36}$$

for a parameterization $\mathcal{C} = \mathcal{C}(\boldsymbol{\phi})$ of $U(K)$ in terms of variables $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_{K^2})$. Here we denote $\boldsymbol{C} = (C_{1\,K+1}, C_{1\,K+2}, \ldots, C_{K\,2K})$. Recall the notation for the components of $\mathcal{C}$ from (1.41). In order to obtain the Haar measure, we assume $\mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_{K^2}$ to be positively oriented and thus replace it by the measure $\mathrm{d}\phi_1 \cdots \mathrm{d}\phi_{K^2}$.

Let us continue by discussing some examples. The formulas stated here will be employed in subsequent sections for the computation of sample Yangian invariants. We parameterize $U(1)$ as

$$\mathcal{C} = C_{12} = e^{i\gamma} \quad \text{with} \quad \gamma \in [0, 2\pi]\,. \tag{4.37}$$

The Haar measure is given by

$$[\mathrm{d}\mathcal{C}] = \chi_1\, i\, \mathrm{d}\gamma \tag{4.38}$$

with $\chi_1 = -i(2\pi)^{-1}$. The group $U(2) = SU(2) \rtimes U(1)$ is parameterized as

$$\begin{aligned}\mathcal{C} = \begin{pmatrix} C_{13} & C_{14} \\ C_{23} & C_{24}\end{pmatrix} &= \begin{pmatrix} e^{i\alpha}\cos\theta & -e^{i\beta}\sin\theta \\ e^{-i\beta}\sin\theta & e^{-i\alpha}\cos\theta\end{pmatrix}\begin{pmatrix} e^{i\gamma} & 0 \\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix} e^{i(\gamma+\alpha)}\cos\theta & -e^{i\beta}\sin\theta \\ e^{i(\gamma-\beta)}\sin\theta & e^{-i\alpha}\cos\theta\end{pmatrix}\end{aligned} \tag{4.39}$$

with

$$\alpha, \beta, \gamma \in [0, 2\pi], \quad \theta \in [0, \tfrac{\pi}{2}]\,. \tag{4.40}$$

The Haar measure is

$$[\mathrm{d}\mathcal{C}] = \chi_2\, i \sin(2\theta)\, \mathrm{d}\theta\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma \tag{4.41}$$

with $\chi_2 = -i(2\pi)^{-3}$. A parameterization of $U(3) = SU(3) \rtimes U(1)$ is

$$\begin{aligned}\mathcal{C} = \begin{pmatrix} C_{14} & C_{15} & C_{16} \\ C_{24} & C_{25} & C_{26} \\ C_{34} & C_{35} & C_{36} \end{pmatrix} = & \begin{pmatrix} e^{i\alpha_1}\cos\theta_1 & -e^{i\beta_1}\sin\theta_1 & 0 \\ e^{-i\beta_1}\sin\theta_1 & e^{-i\alpha_1}\cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ & \cdot \begin{pmatrix} e^{i\alpha_2}\cos\theta_2 & 0 & -e^{i\beta_2}\sin\theta_2 \\ 0 & 1 & 0 \\ e^{-i\beta_2}\sin\theta_2 & 0 & e^{-i\alpha_2}\cos\theta_2 \end{pmatrix} \\ & \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_3}\cos\theta_3 & -e^{i\beta_3}\sin\theta_3 \\ 0 & e^{-i\beta_3}\sin\theta_3 & e^{-i\alpha_3}\cos\theta_3 \end{pmatrix} \begin{pmatrix} e^{i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned} \tag{4.42}$$

Here the $SU(3)$ part is given in terms of $SU(2)$ matrices and

$$\alpha_1, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma \in [0, 2\pi], \quad \alpha_2 = 0, \quad \theta_1, \theta_2, \theta_3 \in [0, \tfrac{\pi}{2}]\,. \tag{4.43}$$

More explicitly the parameterization (4.42) reads

$$\mathcal{C} = \begin{pmatrix} e^{i(\gamma+\alpha_1)}\cos\theta_1\cos\theta_2 & C_{15} & C_{16} \\ e^{i(\gamma-\beta_1)}\sin\theta_1\cos\theta_2 & C_{25} & C_{26} \\ e^{i(\gamma-\beta_2)}\sin\theta_2 & e^{-i\beta_3}\cos\theta_2\sin\theta_3 & e^{-i\alpha_3}\cos\theta_2\cos\theta_3 \end{pmatrix} \tag{4.44}$$

with

$$\begin{aligned} C_{15} &= -e^{i(\beta_1+\alpha_3)}\sin\theta_1\cos\theta_3 - e^{i(\alpha_1+\beta_2-\beta_3)}\cos\theta_1\sin\theta_2\sin\theta_3\,, \\ C_{16} &= e^{i(\beta_1+\beta_3)}\sin\theta_1\sin\theta_3 - e^{i(\alpha_1+\beta_2-\alpha_3)}\cos\theta_1\sin\theta_2\cos\theta_3\,, \\ C_{25} &= e^{i(-\alpha_1+\alpha_3)}\cos\theta_1\cos\theta_3 - e^{i(-\beta_1+\beta_2-\beta_3)}\sin\theta_1\sin\theta_2\sin\theta_3\,, \\ C_{26} &= -e^{i(-\alpha_1+\beta_3)}\cos\theta_1\sin\theta_3 - e^{i(-\beta_1+\beta_2-\alpha_3)}\sin\theta_1\sin\theta_2\cos\theta_3\,. \end{aligned} \tag{4.45}$$

The Haar measure is given by

$$[\mathrm{d}\mathcal{C}] = \chi_3\,(\cos\theta_2)^2\sin(2\theta_1)\sin(2\theta_2)\sin(2\theta_3)\,\mathrm{d}\theta_1\,\mathrm{d}\alpha_1\,\mathrm{d}\beta_1\,\mathrm{d}\theta_2\,\mathrm{d}\beta_2\,\mathrm{d}\theta_3\,\mathrm{d}\alpha_3\,\mathrm{d}\beta_3\,\mathrm{d}\gamma \tag{4.46}$$

with $\chi_3 = 2(2\pi)^{-6}$. Let us at this point refer the reader to appendix B.1. There the parameterization of the $U(3)$ contour in (4.42), which is associated with the invariant $|\Psi_{6,3}\rangle$, emerges naturally by gluing three invariants of the type $|\Psi_{4,2}\rangle$, each of which is obtained from a $U(2)$ integral.

4.1.5 Sample Invariants

Currently, there are no efficient techniques available to evaluate our unitary Graßmannian matrix model (4.24) with the integrand (4.30) for the Yangian invariant $|\Psi_{2K,K}\rangle$ in full generality. In particular, there is no analogue of the formula (4.23), which applies to the special case where our model reduces to the Leutwyler-Smilga integral. Thus we study (4.24) by evaluating it "by hand" for the simplest sample invariants $|\Psi_{2,1}\rangle$, $|\Psi_{4,2}\rangle$ and $|\Psi_{6,3}\rangle$. For these computations we make use of the parameterizations of the unitary contour and the formulas for the Haar measure from section 4.1.4.2.

4.1.5.1 Two-Site Invariant

We evaluate the Graßmannian integral (4.24) for the two-site invariant $|\Psi_{2,1}\rangle$ with representations of the non-compact superalgebra $\mathfrak{u}(p+q|r+s)$. With the integrand (4.31), the parameterization (4.37) of $U(1)$ and the Haar measure (4.38), we obtain

$$
\begin{aligned}
|\Psi_{2,1}\rangle &= 2\pi i \sum_{\substack{g_{12},h_{12}=0\\ g_{12}-h_{12}=q-s-c_1}}^{\infty} \frac{(1\bullet 2)^{g_{12}}}{g_{12}!}\frac{(1\circ 2)^{h_{12}}}{h_{12}!}|0\rangle \\
&= 2\pi i\, \frac{I_{q-s-c_1}(2\sqrt{(1\bullet 2)(1\circ 2)})}{\sqrt{(1\bullet 2)(1\circ 2)}^{\,q-s-c_1}}(1\bullet 2)^{q-s-c_1}|0\rangle\,.
\end{aligned}
\tag{4.47}
$$

To derive this result we treated the $U(1)$ integral in the variable γ, cf. (4.37), as a complex contour integral in $e^{i\gamma}$ and applied the residue theorem. Note that the integrand (4.31) of (4.24) for $|\Psi_{2,1}\rangle$ only contains a factor $\det\mathcal{C}$. Thus (4.24) for $|\Psi_{2,1}\rangle$ is a Leutwyler-Smilga integral (4.22) even for general deformation parameters $v_i^{\pm}$. We presented the formula (4.47) for $|\Psi_{2,1}\rangle$, up to the choice of the normalization, already in section 2.4.2.1. Here we saw how this most simple non-compact Yangian invariant originates from the general unitary Graßmannian integral (4.24).

4.1.5.2 Four-Site Invariant

We continue with the evaluation of the integral (4.24) for the invariant $|\Psi_{4,2}\rangle$ in case of the algebra $\mathfrak{u}(p,q|r+s)$. Its integrand can be found in (4.32). We use the parameterization (4.39) of $U(2)$ with the Haar measure (4.41). The integrals in the variables $e^{i\alpha}, e^{i\beta}$ and $e^{i\gamma}$ are performed using the residue theorem. The remaining integral in θ then reduces to the Euler beta function,

$$
\mathrm{B}(x,y) = 2\int_0^{\frac{\pi}{2}} \mathrm{d}\theta (\sin\theta)^{2x-1}(\cos\theta)^{2y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\tag{4.48}
$$

for $\operatorname{Re} x, \operatorname{Re} y > 0$, cf. [149]. This leads to the invariant

$$
\begin{aligned}
|\Psi_{4,2}\rangle = -(-1)^{c_1+c_2}(2\pi i)^3 &\sum_{\substack{g_{13},\ldots,g_{24}=0\\ h_{13},\ldots,h_{24}=0\\ \text{with (4.50)}}}^{\infty} \frac{(1\bullet 3)^{g_{13}}}{g_{13}!}\frac{(1\bullet 4)^{g_{14}}}{g_{14}!}\frac{(2\bullet 3)^{g_{23}}}{g_{23}!}\frac{(2\bullet 4)^{g_{24}}}{g_{24}!} \\
&\cdot \frac{(1\circ 3)^{h_{13}}}{h_{13}!}\frac{(1\circ 4)^{h_{14}}}{h_{14}!}\frac{(2\circ 3)^{h_{23}}}{h_{23}!}\frac{(2\circ 4)^{h_{24}}}{h_{24}!}|0\rangle \\
&\cdot (-1)^{g_{14}+h_{14}}\mathrm{B}(g_{14}+h_{23}+1, h_{13}+g_{24}-v_1+v_2)\,.
\end{aligned}
\tag{4.49}
$$

In this formula the summation range is constrained by

$$
\begin{aligned}
g_{13}-h_{13}+g_{14}-h_{14} &= -c_1+q-s\,, & g_{23}-h_{23}+g_{24}-h_{24} &= -c_2+q-s\,,\\
g_{13}-h_{13}+g_{23}-h_{23} &= c_3+q-s\,, & g_{14}-h_{14}+g_{24}-h_{24} &= c_4+q-s\,.
\end{aligned}
\tag{4.50}
$$

Furthermore, we have to assume $\operatorname{Re}(v_2-v_1)>0$ in order for the beta function integral to converge. We displayed the expression (4.49) for $|\Psi_{4,2}\rangle$ with a different normalization already above in section 2.4.2.3 without giving a derivation nor showing its Yangian invariance. These gaps are filled now. Recall also from this section that the invariant $|\Psi_{4,2}\rangle$ is of special importance because it is equivalent to an R-matrix. Moreover, this invariant is the first case where the unitary Graßmannian integral (4.24) goes beyond the Leutwyler-Smilga model (4.22) because the integrand (4.32) contains more than just a factor of $\det\mathcal{C}$.

4.1.5.3 Six-Site Invariant

For the computation of the invariant $|\Psi_{6,3}\rangle$ from the Graßmannian integral (4.24) we restrict for simplicity to the compact algebra $\mathfrak{u}(p)$, i.e. $\mathbf{I}_\circ = 0$. The integrand is given in (4.33). We use the parameterization (4.42) of $U(3)$ with the Haar measure (4.46). In the compact case the representation labels satisfy $c_4 = -c_1, c_5 = -c_2, c_6 = -c_3 \geq 0$. The integral is addressed using the residue theorem for the integration variables $e^{i\alpha_1}$, $e^{i\alpha_3}$, $e^{i\beta_1}$, $e^{i\beta_2}$, $e^{i\beta_3}$ and $e^{i\gamma}$. The remaining integrals in θ_1, θ_2 and θ_3 then reduce to Euler beta functions, cf. (4.48). In this way we obtain

$$
\begin{aligned}
|\Psi_{6,3}\rangle = -(2\pi i)^6 \sum_{\substack{k_{14},k_{15},k_{24},k_{25}=0\\ l_{15},l_{16},l_{25}=0}}^{\infty} & \frac{(1\bullet 4)^{k_{14}}}{k_{14}!}\frac{(1\bullet 5)^{k_{15}}}{k_{15}!}\frac{(1\bullet 6)^{c_4-k_{14}-k_{15}}}{(c_4-k_{14}-k_{15})!}\\
&\cdot \frac{(2\bullet 4)^{k_{24}}}{k_{24}!}\frac{(2\bullet 5)^{k_{25}}}{k_{25}!}\frac{(2\bullet 6)^{c_5-k_{24}-k_{25}}}{(c_5-k_{24}-k_{25})!}\\
&\cdot \frac{(3\bullet 4)^{c_4-k_{14}-k_{24}}}{(c_4-k_{14}-k_{24})!}\frac{(3\bullet 5)^{c_5-k_{15}-k_{25}}}{(c_5-k_{15}-k_{25})!}\frac{(3\bullet 6)^{-c_4-c_5+c_6+k_{14}+k_{15}+k_{24}+k_{25}}}{(-c_4-c_5+c_6+k_{14}+k_{15}+k_{24}+k_{25})!}|0\rangle\\
&\cdot \binom{k_{15}}{l_{15}}\binom{c_4-k_{14}-k_{15}}{l_{16}}\binom{k_{25}}{l_{25}}\binom{c_5-k_{24}-k_{25}}{-c_4+c_5+k_{14}+k_{15}-l_{15}+l_{16}-l_{25}}\\
&\cdot (-1)^{c_5+k_{15}+k_{24}+l_{16}+l_{25}}\mathrm{B}(1+c_4-k_{14}-k_{24}, -c_4+c_6+k_{14}+k_{24}-v_1+v_3)\\
&\cdot \mathrm{B}(1+c_4-k_{14}-k_{15}+l_{15}-l_{16}, -c_4+c_5+k_{14}+k_{15}-l_{15}+l_{16}-v_1+v_2)\\
&\cdot \mathrm{B}(1+c_5-l_{15}-l_{25}, -c_5+c_6+l_{15}+l_{25}-v_2+v_3)\,.
\end{aligned}
\tag{4.51}
$$

Here we expressed some combinatorial factors as binomial coefficients. Furthermore, we have to assume $-c_5 + c_6 > \mathrm{Re}\,(v_2 - v_3)$ and $-c_4 + c_6 > \mathrm{Re}\,(v_1 - v_3)$ for the beta function integrals to converge. Note that the infinite sums truncate to finite ones due to the factorials. Hence the invariant is a polynomial in the oscillator contractions $(k \bullet l)$. We checked the Yangian invariance of (4.51) also independently of the proof in section 4.1.2 using computer algebra for small values of the representation labels. The complicated structure of the formula (4.51) emphasizes the need for a more efficient method to evaluate the unitary Graßmannian matrix model (4.24). However, this route is not pursued further in this thesis.

4.2 From Oscillators to Spinor Helicity Variables

So far we investigated the unitary Graßmannian matrix model (4.24) for Yangian invariants with oscillator representations of $\mathfrak{u}(p,q|m)$. In particular, this includes representations of the superconformal algebra $\mathfrak{psu}(2,2|4)$. As we reviewed in the introductory section 1.3, tree-level scattering amplitudes of $\mathcal{N} = 4$ SYM are Yangian invariants with certain representations of this algebra. This raises the question how the invariants computed by (4.24) are related to these amplitudes. We address it in the following by applying a change of basis to the oscillators of $\mathfrak{u}(2,2|4)$ that turns them into the spinor helicity variables of section 1.3. In fact, we implement this basis transformation to spinor helicity-like variables more generally for $\mathfrak{u}(p,p|m)$.

We proceed in several steps. In section 4.2.1 we introduce the Bargmann transformation. This integral transformation is known from the one-dimensional quantum mechanical harmonic oscillator. There it essentially relates the Fock states to the wave functions in position space. We apply this transformation in section 4.2.2 to express the generators of the bosonic $\mathfrak{u}(p,p)$ oscillator representations in terms of spinor helicity-like variables. In section 4.2.3 we comment on the resulting form of the Lax operators, that contain these generators. The Bargmann transformation is applied to the integrand of the Graßmannian matrix model (4.24) for $\mathfrak{u}(p,p)$ in section 4.2.4. Finally, we work out the extension of these calculations to the superalgebra $\mathfrak{u}(p,p|m)$ in section 4.2.5. This material provides the necessary groundwork for addressing the computation of $\mathcal{N}=4$ SYM amplitudes by means of a Graßmannian integral with a unitary contour in section 4.3.

4.2.1 Bargmann Transformation

We introduce the Bargmann transformation along the lines of the original publication [195]. From the outset, we work in a multi-dimensional setting because it is needed for our application of the transformation later on. All formulas straightforwardly reduce to the one-dimensional case, where they describe the simple harmonic oscillator in quantum mechanics. At times we employ this example to provide some intuition for key equations.

We start out with a family of bosonic oscillators on a Fock space obeying

$$[\mathbf{A}_{\mathcal{A}}, \bar{\mathbf{A}}_{\mathcal{B}}] = \delta_{\mathcal{A}\mathcal{B}}\,, \quad {\mathbf{A}_{\mathcal{A}}}^\dagger = \bar{\mathbf{A}}_{\mathcal{A}}\,, \quad \mathbf{A}_{\mathcal{A}}|0\rangle = 0 \tag{4.52}$$

with $\mathcal{A},\mathcal{B} = 1,\ldots,n$. Let $\mathbf{A} = (\mathbf{A}_{\mathcal{A}})$ etc. denote an n-component column vector. The relations in (4.52) are realized by the *Bargmann representation*

$$\bar{\mathbf{A}} \mapsto z\,, \quad \mathbf{A} \mapsto \partial_z\,, \quad |0\rangle \mapsto \Psi_0(z) = 1 \tag{4.53}$$

on the Bargmann space $\mathcal{H}_{\mathrm{B}}$. This is the Hilbert space of holomorphic functions of $z \in \mathbb{C}^n$ with the inner product

$$\langle \Psi(z), \Phi(z)\rangle_{\mathrm{B}} = \int_{\mathbb{C}^n} \frac{\mathrm{d}^n\overline{z}\,\mathrm{d}^n z}{(2\pi i)^n} e^{-\overline{z}^t z}\,\overline{\Psi(z)}\Phi(z)\,, \tag{4.54}$$

where $(2i)^{-n}\,\mathrm{d}^n\overline{z}\,\mathrm{d}^n z = \mathrm{d}^n\mathrm{Re}\,z\,\mathrm{d}^n\mathrm{Im}\,z$ is understood as the measure on $\mathbb{R}^{2n}$. In particular, this inner product implements the reality condition in (4.52), i.e. ${\partial_{z_{\mathcal{A}}}}^\dagger = z_{\mathcal{A}}$. The Bargmann representation can be thought of as a concrete realization of the formal Fock space operators. For recent expositions of this representation see also e.g. [196, 197], where it is, however, called “holomorphic representation”.

In addition, we introduce another family of canonical variables obeying different reality conditions,

$$[\partial_{x_{\mathcal{A}}}, x_{\mathcal{B}}] = \delta_{\mathcal{A}\mathcal{B}}\,, \quad {\partial_{x_{\mathcal{A}}}}^\dagger = -\partial_{x_{\mathcal{A}}}\,, \quad {x_{\mathcal{A}}}^\dagger = x_{\mathcal{A}}\,. \tag{4.55}$$

These are considered as operators on the Hilbert space $\mathcal{H}_{\mathrm{Sch}}$ of square integrable functions of the variable $x \in \mathbb{R}^n$ with the inner product

$$\langle \Psi(x), \Phi(x)\rangle_{\mathrm{Sch}} = \int_{\mathbb{R}^n} \mathrm{d}^n x\,\overline{\Psi(x)}\Phi(x)\,. \tag{4.56}$$

This realization of (4.55) is referred to as *Schrödinger representation*. For the example of the one-dimensional harmonic oscillator, this may be interpreted as the realization in position space.

We observe that by a naive counting the degrees of freedom in $\mathcal{H}_{\mathrm{B}}$ and $\mathcal{H}_{\mathrm{Sch}}$ do match. A function $\Psi(z)$ in $\mathcal{H}_{\mathrm{B}}$ depends on n complex coordinates $z_{\mathcal{A}}$ but not on their conjugates

$\bar{z}_{\mathcal{A}}$. Similarly, $\Psi(x)$ in $\mathcal{H}_{\rm Sch}$ is a function of n real coordinates $x_{\mathcal{A}}$. Thus we want to identify the canonical variables in $\mathcal{H}_{\rm B}$ and $\mathcal{H}_{\rm Sch}$. For this purpose we make the ansatz

$$\partial_z \leftrightarrow Ax + B\partial_x\,, \quad z \leftrightarrow \overline{A}x - \overline{B}\partial_x\,, \tag{4.57}$$

where we allow for $n \times n$ matrices A, B and their complex conjugates $\overline{A}, \overline{B}$. The latter relation is obtained from the first one by taking the Hilbert space adjoint. For (4.57) to be compatible with the commutation relations and reality conditions in (4.52) and (4.55) we have to impose

$$AB^\dagger + BA^\dagger = 1_n\,, \quad BA^t = AB^t\,, \tag{4.58}$$

where † stands for Hermitian conjugation and ^{t} for transposition of matrices. From now on we concentrate for simplicity on the special class of solutions of (4.58) where

$$2\gamma\, AA^\dagger = 1_n\,, \quad B = \gamma A \quad \text{with} \quad \gamma \in \mathbb{R}\,. \tag{4.59}$$

Note that this condition can be solved trivially by taking $A \propto 1_n$, in which case the components of the relations in (4.57) decouple. The identification (4.57) of the Hilbert spaces $\mathcal{H}_{\rm B}$ and $\mathcal{H}_{\rm Sch}$ is implemented by the *Bargmann transformation*

$$\Psi(z) = \langle\overline{\mathcal{K}(z,x)}, \Psi(x)\rangle_{\rm Sch}\,, \quad \Psi(x) = \langle\mathcal{K}(z,x), \Psi(z)\rangle_{\rm B} \tag{4.60}$$

with the kernel

$$\mathcal{K}(z,x) = (\pi\gamma)^{-\frac{n}{4}} e^{-\gamma z^t AA^t z - \frac{1}{2\gamma}x^t x + 2z^t Ax}\,. \tag{4.61}$$

This kernel solves the differential equations obtained by imposing (4.57) on (4.60),

$$\partial_z\mathcal{K}(z,x) = A(x - \gamma\partial_x)\mathcal{K}(z,x)\,, \quad z\mathcal{K}(z,x) = \overline{A}(x + \gamma\partial_x)\mathcal{K}(z,x)\,. \tag{4.62}$$

The prefactor in (4.61) is fixed by demanding that the transformation (4.60) preserves the norm of the vacuum state, $\|\Psi_0(z)\|_{\rm B} = \|\Psi_0(x)\|_{\rm Sch} = 1$, where

$$\Psi_0(x) = (\pi\gamma)^{-\frac{n}{4}} e^{-\frac{1}{2\gamma}x^t x}\,. \tag{4.63}$$

The Bargmann transformation is unitary, i.e.

$$\langle\Psi(z), \Phi(z)\rangle_{\rm B} = \langle\Psi(x), \Phi(x)\rangle_{\rm Sch}\,, \tag{4.64}$$

because an orthonormal basis of both Hilbert spaces can be built by acting with the canonical variables on the vacuum. In the example of the one-dimensional quantum mechanical oscillator, a Bargmann transformation like (4.60) implements the change of basis between the Fock (or rather Bargmann) and the position space (Schrödinger) representation.

4.2.2 Transformation of Bosonic Non-Compact Generators

Here we express the generators of the $\mathfrak{u}(p,p)$ oscillator representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_c$ from section 2.3 in terms of spinor helicity-like variables. An analogous calculation in the special case of the conformal algebra $\mathfrak{su}(2,2)$ can be found in [150], see also [151].[2]

[2]The so-called ladder representations of the conformal algebra $\mathfrak{su}(2,2)$ in terms of spinor helicity variables from [151] can be exponentiated to representations of the Lie group $SU(2,2)$. This gives rise to transformations laws under *finite* conformal transformations [198]. The resulting formulas are also known in the mathematical literature, see e.g. [199, 200]. Our discussion here is confined to the *infinitesimal* Lie algebra representations.

We identify the oscillators $\bar{\mathbf{A}}_{\mathcal{A}}$ of the previous section, and therefore the Bargmann variables $z_{\mathcal{A}}$, with those in the generators of the $\mathfrak{u}(p,p)$ representations from section 2.3. Thus we choose the number of oscillators in (4.52) to be $n = 2p$. Likewise, we want to relate the Schrödinger variables $x_{\mathcal{A}}$ to analogues of the spinor helicity variables that we employed for the description of $\mathcal{N} = 4$ SYM amplitudes in section 1.3. Recall that those spinor helicity variables are complex. Thus to establish the relation we have to introduce complex coordinates in $\mathbb{R}^{2p}$ by

$$\begin{pmatrix} \sigma_\alpha \\ \hdashline \overline{\sigma}_\alpha \end{pmatrix} = \left(\begin{array}{c:c} 1_p & i1_p \\ \hdashline 1_p & -i1_p \end{array}\right) \begin{pmatrix} x_{\mathsf{A}} \\ \hdashline x_{\dot{\mathsf{A}}} \end{pmatrix}, \qquad \begin{pmatrix} \partial_{\sigma_\alpha} \\ \hdashline \partial_{\overline{\sigma}_\alpha} \end{pmatrix} = \frac{1}{2} \left(\begin{array}{c:c} 1_p & -i1_p \\ \hdashline 1_p & i1_p \end{array}\right) \begin{pmatrix} \partial_{x_{\mathsf{A}}} \\ \hdashline \partial_{x_{\dot{\mathsf{A}}}} \end{pmatrix}. \tag{4.65}$$

Here we split $x = (x_{\mathcal{A}}) \in \mathbb{R}^{2p}$ into its components x_{A} with $\mathsf{A} = 1, \dots, p$ and $x_{\dot{\mathsf{A}}}$ with $\dot{\mathsf{A}} = p+1, \dots, 2p$. The new coordinates are $\sigma = (\sigma_\alpha) \in \mathbb{C}^p$, i.e. $\alpha = 1, \dots, p$, and its complex conjugate $\overline{\sigma}$. See also (2.32) and the text before and after that equation for explanations on the index ranges. Employing these variables, the properties (4.55) of the operators in $\mathcal{H}_{\mathrm{Sch}}$ read

$$[\partial_{\sigma_\alpha}, \sigma_\beta] = \delta_{\alpha\beta}, \quad {\sigma_\alpha}^\dagger = \overline{\sigma}_\alpha, \quad {\partial_{\sigma_\alpha}}^\dagger = -\partial_{\overline{\sigma}_\alpha}. \tag{4.66}$$

The inner product (4.56) becomes

$$\langle \Psi(\sigma, \overline{\sigma}), \Phi(\sigma, \overline{\sigma}) \rangle_{\mathrm{Sch}} = \int_{\mathbb{C}^p} \frac{\mathrm{d}^p \overline{\sigma}\, \mathrm{d}^p \sigma}{(2i)^p} \overline{\Psi(\sigma, \overline{\sigma})} \Phi(\sigma, \overline{\sigma}), \tag{4.67}$$

where the measure on $\mathbb{C}^p$ is defined as in the context of (4.54). Furthermore, we select the particular solution $A = \frac{1}{\sqrt{2}} \overline{E}$ and $\gamma = \frac{1}{2}$ of (4.59). This transforms the relation between the operators in $\mathcal{H}_{\mathrm{B}}$ and $\mathcal{H}_{\mathrm{Sch}}$ from (4.57) and the vacuum state in (4.63) into

$$\begin{gathered} \begin{pmatrix} z_{\mathsf{A}} \\ \hdashline z_{\dot{\mathsf{A}}} \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_\alpha - \partial_{\overline{\sigma}_\alpha} \\ \hdashline \overline{\sigma}_\alpha - \partial_{\sigma_\alpha} \end{pmatrix}, \qquad \begin{pmatrix} \partial_{z_{\mathsf{A}}} \\ \hdashline \partial_{z_{\dot{\mathsf{A}}}} \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} \overline{\sigma}_\alpha + \partial_{\sigma_\alpha} \\ \hdashline \sigma_\alpha + \partial_{\overline{\sigma}_\alpha} \end{pmatrix}, \\ \Psi_0(\sigma, \overline{\sigma}) = \sqrt{\frac{2}{\pi}}^{\,p} e^{-\overline{\sigma}^t \sigma}. \end{gathered} \tag{4.68}$$

The Bargmann transformation (4.60) mapping $\mathcal{H}_{\mathrm{B}} \to \mathcal{H}_{\mathrm{Sch}}$ becomes explicitly

$$\Psi(\sigma, \overline{\sigma}) = \sqrt{\frac{2}{\pi}}^{\,p} e^{-\overline{\sigma}^t \sigma} \int_{\mathbb{C}^{2p}} \frac{\mathrm{d}^{2p} \overline{z}\, \mathrm{d}^{2p} z}{(2\pi i)^{2p}} e^{-\overline{\dot{z}}^t \dot{z} - \overline{\mathring{z}}^t \mathring{z} - \overline{\mathring{z}}^t \overline{\dot{z}} + \sqrt{2}(\overline{\dot{z}}^t \sigma + \overline{\mathring{z}}^t \overline{\sigma})} \Psi(z), \tag{4.69}$$

where we defined $\dot{z} = (z_{\mathsf{A}}), \mathring{z} = (z_{\dot{\mathsf{A}}}) \in \mathbb{C}^p$. The bullet and the circle in this notation are in analogy to those of the oscillator contractions in (4.3). The integral transformation (4.69) will we be crucial in section 4.2.4 because it allows us to express the unitary Graßmannian matrix model (4.24) in terms of spinor helicity-like variables.

We turn our attention to the generators of the "ordinary" oscillator representation $\mathcal{D}_c$ and the "dual" one $\bar{\mathcal{D}}_c$, whose Fock space realizations can be found in (2.34) and (2.36), respectively. For their realization in the Bargmann space $\mathcal{H}_{\mathrm{B}}$, we use the same symbol as in the Fock space and denote them by $\mathbf{J}_{\mathcal{A}\mathcal{B}}$ at ordinary and by $\bar{\mathbf{J}}_{\mathcal{A}\mathcal{B}}$ at dual sites. In $\mathcal{H}_{\mathrm{Sch}}$ we write $\mathfrak{J}_{\mathcal{A}\mathcal{B}}$ and $\bar{\mathfrak{J}}_{\mathcal{A}\mathcal{B}}$, respectively. To match with the spinor helicity variables later on, we have to redefine the variables for $\mathcal{H}_{\mathrm{Sch}}$ once more depending on the type of site by introducing

$$\lambda = \begin{cases} \sigma & \text{for ordinary sites}, \\ -\overline{\sigma} & \text{for dual sites}. \end{cases} \tag{4.70}$$

From (4.68) we then obtain

$$
\begin{aligned}
(\mathbf{J}_{\mathcal{AB}}) &= \left(\begin{array}{c:c} z_{\mathsf{A}}\partial_{z_{\mathsf{B}}} & z_{\mathsf{A}}z_{\dot{\mathsf{B}}} \\ \hdashline -\partial_{z_{\dot{\mathsf{A}}}}\partial_{z_{\mathsf{B}}} & -\partial_{z_{\dot{\mathsf{A}}}}z_{\dot{\mathsf{B}}} \end{array}\right) \quad \leftrightarrow \quad (\mathfrak{J}_{\mathcal{AB}}) = D \left(\begin{array}{c:c} \lambda_\alpha\partial_{\lambda_\beta} & \lambda_\alpha\overline{\lambda}_\beta \\ \hdashline -\partial_{\overline{\lambda}_\alpha}\partial_{\lambda_\beta} & -\partial_{\overline{\lambda}_\alpha}\overline{\lambda}_\beta \end{array}\right) D^{-1}, \\
(\bar{\mathbf{J}}_{\mathcal{AB}}) &= \left(\begin{array}{c:c} -z_{\mathsf{B}}\partial_{z_{\mathsf{A}}} & -\partial_{z_{\dot{\mathsf{B}}}}\partial_{z_{\mathsf{A}}} \\ \hdashline z_{\mathsf{B}}z_{\dot{\mathsf{A}}} & \partial_{z_{\dot{\mathsf{B}}}}z_{\dot{\mathsf{A}}} \end{array}\right) \quad \leftrightarrow \quad (\bar{\mathfrak{J}}_{\mathcal{AB}}) = D \left(\begin{array}{c:c} \partial_{\lambda_\beta}\lambda_\alpha & -\overline{\lambda}_\beta\lambda_\alpha \\ \hdashline \partial_{\lambda_\beta}\partial_{\overline{\lambda}_\alpha} & -\overline{\lambda}_\beta\partial_{\overline{\lambda}_\alpha} \end{array}\right) D^{-1}
\end{aligned}
\tag{4.71}
$$

with

$$
D = \left(\begin{array}{c:c} 1_p & 1_p \\ \hdashline -1_p & 1_p \end{array}\right). \tag{4.72}
$$

Notice that in $\mathcal{H}_{\mathrm{B}}$ the form of the $\mathfrak{u}(p,p)$ generators $\mathbf{J}_{\mathcal{AB}}$ at ordinary sites and $\bar{\mathbf{J}}_{\mathcal{AB}}$ at dual sites differs “considerably”. For example the upper right block contains two coordinates for the former generators and two derivatives for the latter. In contrast, the generators look “almost alike” in $\mathcal{H}_{\mathrm{Sch}}$,

$$
\bar{\mathfrak{J}}_{\mathcal{AB}}\Big|_{(\lambda,\overline{\lambda})\mapsto(\lambda,-\overline{\lambda})} = \mathfrak{J}_{\mathcal{AB}} + \delta_{\mathcal{AB}}\,. \tag{4.73}
$$

The central elements (2.35) and (2.37) that function as representation labels become, respectively,

$$
\begin{aligned}
\mathbf{C} = \mathrm{tr}(\mathbf{J}_{\mathcal{AB}}) \leftrightarrow \mathfrak{C} &= \sum_{\alpha=1}^{p}(\lambda_\alpha\partial_{\lambda_\alpha} - \overline{\lambda}_\alpha\partial_{\overline{\lambda}_\alpha}) - p\,, \\
\bar{\mathbf{C}} = \mathrm{tr}(\bar{\mathbf{J}}_{\mathcal{AB}}) \leftrightarrow \bar{\mathfrak{C}} &= \sum_{\alpha=1}^{p}(\lambda_\alpha\partial_{\lambda_\alpha} - \overline{\lambda}_\alpha\partial_{\overline{\lambda}_\alpha}) + p\,.
\end{aligned}
\tag{4.74}
$$

At this point we can compare the $\mathfrak{u}(2,2)$ case of the generators in (4.71) with the bosonic part of the generators (1.30) for scattering amplitudes, which are expressed in terms of spinor helicity variables. We disregard the similarity transformation with the matrix D in (4.71) for this comparison, see, however, section 4.2.3 below. Under these premises the generators $\mathfrak{J}_{\mathcal{AB}}$ of the representation $\mathcal{D}_c$ in (4.71) agree with those in (1.30) after setting $(\tilde{\lambda}_{\dot{\alpha}}) = +(\overline{\lambda}_\alpha)$. The generators $\bar{\mathfrak{J}}_{\mathcal{AB}}$ of $\bar{\mathcal{D}}_c$ in (4.71) match those in (1.30) with $(\tilde{\lambda}_{\dot{\alpha}}) = -(\overline{\lambda}_\alpha)$ up to a shift as in (4.73). We recall from the definition of the spinor helicity variables around (1.5) that the sign in the relation $\tilde{\lambda} = \pm\overline{\lambda}$ determines the sign of the energy. Therefore the “ordinary” oscillator representations $\mathcal{D}_c$ correspond to positive energies. Analogously representations of the “dual” class $\bar{\mathcal{D}}_c$ are associated with negative energies. This explains why in section 1.3 seemingly only the *one* type of generators (1.30) appears at all legs of the scattering amplitudes, whereas the *two* types of representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_c$ are omnipresent in the main part of this thesis.

Let us add a comment about the just mentioned shift of the generators at the dual sites. It is the reason why we were only able to show the invariance of the amplitudes under a *special* linear Lie superalgebra, and the associated Yangian, in the introductory section 1.3. In the main part of this thesis we can always work with *general* linear Lie (super)algebras because our definition of the dual representation properly incorporates this shift. On a different note, in (1.7) we defined the operator $\mathfrak{h} = \frac{1}{2}\sum_{\alpha=1}^{2}(\overline{\lambda}_\alpha\partial_{\overline{\lambda}_\alpha} - \lambda_\alpha\partial_{\lambda_\alpha})$ that measures the helicity h of a particle. For the gluon amplitudes discussed in section 1.3.2 it may take the values $h = \pm 1$. Using (4.74) we translate h into the $\mathfrak{u}(2,2)$ oscillator representation label c, which is the eigenvalue of the central elements in that equation,

$$\begin{aligned} h = +1 &\Leftrightarrow c = -4\,, \qquad h = -1 \Leftrightarrow c = 0 \qquad \text{for} \quad \mathcal{D}_c\,, \\ h = +1 &\Leftrightarrow c = 0\,, \qquad h = -1 \Leftrightarrow c = +4 \qquad \text{for} \quad \bar{\mathcal{D}}_c\,. \end{aligned} \tag{4.75}$$

4.2.3 Change of Basis in Lax Operators

A quick calculation shows that the map $J_{\mathcal{AB}} \mapsto \tilde{J}_{\mathcal{AB}}$ defined by

$$(J_{\mathcal{AB}}) = D(\tilde{J}_{\mathcal{AB}})D^{-1}\,, \quad \text{i.e.} \quad J_{\mathcal{AB}} = \sum_{\mathcal{C},\mathcal{D}} D_{\mathcal{AC}}\tilde{J}_{\mathcal{CD}}D^{-1}_{\mathcal{DB}} \tag{4.76}$$

with an even $n|m \times n|m$ supermatrix D is an automorphism of the $\mathfrak{gl}(n|m)$ superalgebra (2.3). This allows us to reformulate the Lax operator (2.18) as

$$\begin{aligned} R_{\square\,\mathcal{V}}(u-v) &= f_{\mathcal{V}}(u-v)\left(1 + (u-v)^{-1}\sum_{\mathcal{A},\mathcal{B}} E_{\mathcal{AB}}J_{\mathcal{BA}}(-1)^{|\mathcal{B}|}\right) \\ &= f_{\mathcal{V}}(u-v)\left(1 + (u-v)^{-1}\sum_{\mathcal{C},\mathcal{D}} \tilde{E}_{\mathcal{DC}}\tilde{J}_{\mathcal{CD}}(-1)^{|\mathcal{C}|}\right)\,, \end{aligned} \tag{4.77}$$

where we introduced

$$\tilde{E}_{\mathcal{DC}} = \sum_{\mathcal{A},\mathcal{B}} D^{-1}_{\mathcal{DA}}E_{\mathcal{AB}}D_{\mathcal{BC}} = (D^{-1})^t e_{\mathcal{DC}} D^t\,. \tag{4.78}$$

We can apply this observation to express the matrix elements the monodromy (2.19) in the basis $\tilde{E}_{\mathcal{AB}}$ instead of $E_{\mathcal{AB}}$. Therefore the similarity transformation (4.76) of the $\mathfrak{gl}(n|m)$ generators can be absorbed in a redefinition of the Yangian generators (2.13). This justifies the negligence of such a transformation towards the end of the previous section, where we compared the generators $\mathfrak{J}_{\mathcal{AB}}$ and $\bar{\mathfrak{J}}_{\mathcal{AB}}$ in (4.71) for the $\mathfrak{u}(2,2)$ case with the bosonic part of those for amplitudes in (1.30). We will refer to the superalgebra case of the observation presented here later in section 4.2.5.

4.2.4 Transformation of Bosonic Graßmannian Integrand

The map in (4.68) transforms the generators of the $\mathfrak{u}(p,p)$ oscillator representations into spinor helicity-like variables. Let us now apply the appendant Bargmann transformation (4.69) to the unitary Graßmannian matrix model (4.24) in order to transform it into those variables. We can focus on the transformation of the exponential function in the integrand of (4.24) because it is the only part containing oscillators. In essence, the Bargmann transformation of this exponential function reduces to a multi-dimensional Gaußian integral. The evaluation of this integral yields a delta function of the spinor helicity-like variables. In the following paragraph we state this result in detail. Its proof occupies the rest of this section.

We concentrate on the oscillator-dependent part of the integrand in the Graßmannian matrix model (4.24) for $\mathfrak{u}(p,p)$, which we denote by $|\Phi\rangle$. To be able to apply the Bargmann transformation, we first have to realize it in the space $\mathcal{H}_{\mathrm{B}}$ using the replacement (4.53),

$$|\Phi\rangle = e^{\operatorname{tr}(\mathcal{C}\mathbf{I}^t_\bullet + \mathbf{I}_\circ\mathcal{C}^\dagger)}|0\rangle \mapsto \Phi(\boldsymbol{z})\,, \tag{4.79}$$

where $\boldsymbol{z} = (z^i_{\mathcal{A}})$ with $i = 1, \dots, N = 2K$ and $\mathcal{A} = 1, \dots, 2p$ collectively denotes all complex Bargmann variables. The functional forms of $|\Phi\rangle$ and $\Phi(\boldsymbol{z})$ are identical. Only the oscillator-valued entries of the matrices $\mathbf{I}_\bullet$ and $\mathbf{I}_\circ$ defined in (4.2) and (4.3) get replaced by

$$(k \bullet l) \mapsto \sum_{\mathsf{A}=1}^{p} z^l_{\mathsf{A}} z^k_{\mathsf{A}}\,, \quad (k \circ l) \mapsto \sum_{\dot{\mathsf{A}}=p+1}^{2p} z^l_{\dot{\mathsf{A}}} z^k_{\dot{\mathsf{A}}}\,, \tag{4.80}$$

and, furthermore, $|0\rangle \mapsto 1$. Here $k = 1, \dots, K$ refers to a dual site and $l = K+1, \dots, 2K$ to an ordinary one. The Bargmann transformation (4.69) can now be applied to $\Phi(\boldsymbol{z})$ in (4.79). After relabeling the variables according to (4.70), this yields the expression of the Graßmannian integrand in $\mathcal{H}_{\mathrm{Sch}}$,

$$\Phi(\boldsymbol{z}) \leftrightarrow \Phi(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}) = \delta^{pK}_{\mathbb{C}}(\boldsymbol{\lambda}^{\mathrm{d}} + \mathcal{C}\boldsymbol{\lambda}^{\mathrm{o}}) = \delta^{pK}_{\mathbb{C}}(C\boldsymbol{\lambda})\,. \tag{4.81}$$

Here the spinor helicity-like variables at the dual and ordinary sites are arranged in the $K \times p$ matrices $\boldsymbol{\lambda}^{\mathrm{d}}$ and $\boldsymbol{\lambda}^{\mathrm{o}}$, respectively. The $2K \times p$ matrix $\boldsymbol{\lambda}$ is built from these two matrices,

$$\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}^{\mathrm{d}} \\ \hdashline \boldsymbol{\lambda}^{\mathrm{o}} \end{pmatrix}, \quad \boldsymbol{\lambda}^{\mathrm{d}} = \begin{pmatrix} \lambda^1_1 & \cdots & \lambda^1_p \\ \vdots & & \vdots \\ \lambda^K_1 & \cdots & \lambda^K_p \end{pmatrix}, \quad \boldsymbol{\lambda}^{\mathrm{o}} = \begin{pmatrix} \lambda^{K+1}_1 & \cdots & \lambda^{K+1}_p \\ \vdots & & \vdots \\ \lambda^{2K}_1 & \cdots & \lambda^{2K}_p \end{pmatrix}. \tag{4.82}$$

The $K \times 2K$ matrix C is an element of the Graßmannian $\mathrm{Gr}(2K, K)$ and it contains the unitary $K \times K$ block $\mathcal{C}$, recall (1.41). Notice that the arguments of the delta functions in (4.81) are complex as this applies to the entries of $\boldsymbol{\lambda}$ and C. A complex delta function is defined as the product of the delta function for the real part of the argument times that for the imaginary part, see also (4.91) below. Equation (4.81) contains the form of the Graßmannian integrand in spinor helicity-like variables that we proclaimed already in the introductory paragraph.

Let us set out to prove the result of the Bargmann transformation presented in (4.81). To begin with, we reformulate the r.h.s. of (4.79) as

$$\Phi(\boldsymbol{z}) = \prod_{\alpha=1}^{p} e^{\frac{1}{2}\boldsymbol{z}^t_\alpha \mathtt{C}\, \boldsymbol{z}_\alpha}\,, \tag{4.83}$$

where we introduced $\boldsymbol{z}_\alpha \in \mathbb{C}^{4K}$ and a $4K \times 4K$ matrix $\mathtt{C}$ presented in terms of its $K \times K$ blocks,

$$\boldsymbol{z}_\alpha = \begin{pmatrix} \dot{\boldsymbol{z}}^{\mathrm{d}}_\alpha \\ \hdashline \dot{\boldsymbol{z}}^{\mathrm{o}}_\alpha \\ \hdashline \mathring{\boldsymbol{z}}^{\mathrm{d}}_\alpha \\ \hdashline \mathring{\boldsymbol{z}}^{\mathrm{o}}_\alpha \end{pmatrix} = \begin{pmatrix} (z^k_\alpha) \\ \hdashline (z^l_\alpha) \\ \hdashline (z^k_{\alpha+p}) \\ \hdashline (z^l_{\alpha+p}) \end{pmatrix}, \quad \mathtt{C} = \left(\begin{array}{c:c:c:c} 0 & \mathcal{C} & 0 & 0 \\ \hdashline \mathcal{C}^t & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & (\mathcal{C}^\dagger)^t \\ \hdashline 0 & 0 & \mathcal{C}^\dagger & 0 \end{array}\right). \tag{4.84}$$

Let us explain our notation. Here $\dot{\boldsymbol{z}}^{\mathrm{d}}_\alpha \in \mathbb{C}^K$ contains the variables z^k_α at dual sites with $k = 1, \dots, K$ and $\dot{\boldsymbol{z}}^{\mathrm{o}}_\alpha \in \mathbb{C}^K$ is built from z^l_α at ordinary sites with $l = K+1, \dots, 2K$ etc. We apply the Bargmann transformation (4.69) to all sites of $\Phi(\boldsymbol{z})$ in (4.83). This yields

$$\begin{aligned} \Phi(\boldsymbol{\sigma}, \overline{\boldsymbol{\sigma}}) &= \prod_{\alpha=1}^{p} \Phi_\alpha(\boldsymbol{\sigma}_\alpha, \overline{\boldsymbol{\sigma}}_\alpha) \\ &= \prod_{\alpha=1}^{p} \left(\frac{2}{\pi}\right)^K e^{-\overline{\boldsymbol{\sigma}}^t_\alpha \boldsymbol{\sigma}_\alpha} \int_{\mathbb{C}^{4K}} \frac{\mathrm{d}^{4K}\overline{\boldsymbol{z}}_\alpha\, \mathrm{d}^{4K}\boldsymbol{z}_\alpha}{(2\pi i)^{4K}}\, e^{\frac{1}{2}\left(\begin{smallmatrix} \boldsymbol{z}_\alpha \\ \hdashline \overline{\boldsymbol{z}}_\alpha \end{smallmatrix}\right)^t \mathtt{H} \left(\begin{smallmatrix} \boldsymbol{z}_\alpha \\ \hdashline \overline{\boldsymbol{z}}_\alpha \end{smallmatrix}\right) + \sqrt{2}\, \overline{\boldsymbol{z}}^t_\alpha \left(\begin{smallmatrix} \boldsymbol{\sigma}_\alpha \\ \hdashline \overline{\boldsymbol{\sigma}}_\alpha \end{smallmatrix}\right)}\,, \end{aligned} \tag{4.85}$$

where

$$\boldsymbol{\sigma}_\alpha = \begin{pmatrix} \boldsymbol{\sigma}^{\mathrm{d}}_\alpha \\ \boldsymbol{\sigma}^{\mathrm{o}}_\alpha \end{pmatrix} = \begin{pmatrix} (\sigma^k_\alpha) \\ (\sigma^l_\alpha) \end{pmatrix}, \quad \mathtt{H} = \begin{pmatrix} 0 & \mathcal{C} & 0 & 0 & -1 & 0 & 0 & 0 \\ \mathcal{C}^t & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & (\mathcal{C}^\dagger)^t & 0 & 0 & -1 & 0 \\ 0 & 0 & \mathcal{C}^\dagger & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{4.86}$$

The variables $\boldsymbol{\sigma}^{\mathrm{d}}_\alpha$ and $\boldsymbol{\sigma}^{\mathrm{o}}_\alpha$ are in $\mathbb{C}^K$. In block matrices we abbreviate the unit matrix 1_K by 1. Furthermore, 0 can represent a $K \times K$ block of zeros or a K-dimensional null vector. We observe a factorization the Bargmann transformation of $\Phi(\boldsymbol{z})$ into p Gaußian integrals of $4K$ complex dimensions in (4.85). These integrals can be evaluated after bringing $\mathtt{H}$ into block-diagonal form,

$$\mathtt{H} = \mathtt{V}^t \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \check{\mathtt{H}} & \\ 0 & 0 & & & \end{pmatrix} \mathtt{V}, \quad \check{\mathtt{H}} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & (\mathcal{C}^\dagger)^t & 0 & -1 & 0 \\ 0 & \mathcal{C}^\dagger & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}, \tag{4.87}$$

where the transformation matrix is given by

$$\mathtt{V} = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ -i\sqrt{2} & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\mathcal{C}^\dagger & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{C}^t & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\mathcal{C}^t & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}^\dagger & 0 & 0 & 1 \end{pmatrix}. \tag{4.88}$$

To change the integration variables in (4.85) to ones which are adapted to this block-diagonal form of $\mathtt{H}$, we compute

$$\mathtt{V}\left(\begin{array}{c} \boldsymbol{z}_\alpha \\ \hline \overline{\boldsymbol{z}}_\alpha \end{array}\right) = \left(\begin{array}{c} 2\sqrt{2}\,\mathrm{Re}\,\dot{\boldsymbol{z}}^{\mathrm{d}}_\alpha \\ \hline 2\sqrt{2}\,\mathrm{Im}\,\dot{\boldsymbol{z}}^{\mathrm{d}}_\alpha \\ \hline \dot{\boldsymbol{z}}^{\mathrm{o}}_\alpha - \mathcal{C}^\dagger \overline{\dot{\boldsymbol{z}}}^{\mathrm{d}}_\alpha \\ \hline \mathring{\boldsymbol{z}}^{\mathrm{d}}_\alpha + \overline{\dot{\boldsymbol{z}}}^{\mathrm{d}}_\alpha \\ \hline \mathring{\boldsymbol{z}}^{\mathrm{o}}_\alpha + \mathcal{C}^t \dot{\boldsymbol{z}}^{\mathrm{d}}_\alpha \\ \hline \overline{\dot{\boldsymbol{z}}}^{\mathrm{o}}_\alpha - \mathcal{C}^t \dot{\boldsymbol{z}}^{\mathrm{d}}_\alpha \\ \hline \overline{\mathring{\boldsymbol{z}}}^{\mathrm{d}}_\alpha + \dot{\boldsymbol{z}}^{\mathrm{d}}_\alpha \\ \hline \overline{\mathring{\boldsymbol{z}}}^{\mathrm{o}}_\alpha + \mathcal{C}^\dagger \overline{\dot{\boldsymbol{z}}}^{\mathrm{d}}_\alpha \end{array}\right) = \left(\begin{array}{c} \boldsymbol{x}_\alpha \\ \hline \boldsymbol{y}_\alpha \\ \hline \boldsymbol{w}_\alpha \\ \hline \overline{\boldsymbol{w}}_\alpha \end{array}\right), \quad (\mathtt{V}^{-1})^t \left(\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \boldsymbol{\sigma}_\alpha \\ \hline \overline{\boldsymbol{\sigma}}_\alpha \end{array}\right) = \left(\begin{array}{c} \frac{-i}{\sqrt{2}}\,\mathrm{Im}(\overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha - \mathcal{C}\boldsymbol{\sigma}^{\mathrm{o}}_\alpha) \\ \hline \frac{-i}{\sqrt{2}}\,\mathrm{Re}(\overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha - \mathcal{C}\boldsymbol{\sigma}^{\mathrm{o}}_\alpha) \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \boldsymbol{\sigma}^{\mathrm{o}}_\alpha \\ \hline \overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha \\ \hline \overline{\boldsymbol{\sigma}}^{\mathrm{o}}_\alpha \end{array}\right). \tag{4.89}$$

Expressing (4.85) in terms of the new variables $\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha \in \mathbb{R}^K$ and $\boldsymbol{w}_\alpha \in \mathbb{C}^{3K}$ leads to

$$\Phi_\alpha(\boldsymbol{\sigma}_\alpha, \overline{\boldsymbol{\sigma}}_\alpha) = \frac{\delta_{\mathbb{C}^K}(\overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha - \mathcal{C}\boldsymbol{\sigma}^{\mathrm{o}}_\alpha)}{e^{\overline{\boldsymbol{\sigma}}^t_\alpha \boldsymbol{\sigma}_\alpha}} \int_{\mathbb{C}^{3K}} \frac{\mathrm{d}^{3K}\overline{\boldsymbol{w}}_\alpha\, \mathrm{d}^{3K}\boldsymbol{w}_\alpha}{(2\pi i)^{3K}}\, e^{\frac{1}{2}\left(\begin{array}{c}\boldsymbol{w}_\alpha \\ \hline \overline{\boldsymbol{w}}_\alpha\end{array}\right)^t \check{\mathtt{H}} \left(\begin{array}{c}\boldsymbol{w}_\alpha \\ \hline \overline{\boldsymbol{w}}_\alpha\end{array}\right) + \sqrt{2}\,\overline{\boldsymbol{w}}^t_\alpha \left(\begin{array}{c}\boldsymbol{\sigma}^{\mathrm{o}}_\alpha \\ \hline \overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha \\ \hline \overline{\boldsymbol{\sigma}}^{\mathrm{o}}_\alpha\end{array}\right)}. \tag{4.90}$$

Here the integrals over $\boldsymbol{x}_\alpha$ and $\boldsymbol{y}_\alpha$, which are associated with the vanishing diagonal blocks of $\mathtt{H}$ in (4.87), reduce to Fourier representations of delta functions. These are defined by

$$\int_{\mathbb{R}^K} \mathrm{d}^K\boldsymbol{x}_\alpha\, e^{-i\boldsymbol{x}^t_\alpha \boldsymbol{\theta}} = (2\pi)^K \delta^K(\boldsymbol{\theta})\,, \quad \delta^K_{\mathbb{C}}(\boldsymbol{\theta} + i\boldsymbol{\varphi}) = \delta^K(\boldsymbol{\theta})\delta^K(\boldsymbol{\varphi}) \tag{4.91}$$

for $\boldsymbol{\theta}, \boldsymbol{\varphi} \in \mathbb{R}^K$. Recall that the symbol δ denotes an ordinary delta function of a real argument. Next, we focus on the evaluation of the remaining integral in (4.90). Recall the standard result on multi-dimensional Gaußian integrals,

$$\int_{\mathbb{R}^n} \mathrm{d}^n\mathtt{u}\, e^{-\frac{1}{2}\mathtt{u}^t\mathcal{A}\mathtt{u} + \mathtt{b}^t\mathtt{u}} = (2\pi)^{\frac{n}{2}} \sqrt{\det\mathcal{A}}^{\,-1} e^{\frac{1}{2}\mathtt{b}^t\mathcal{A}^{-1}\mathtt{b}}\,, \tag{4.92}$$

where $\mathcal{A}$ is a symmetric complex $n \times n$ matrix whose eigenvalues have a strictly positive real part and $\mathtt{b} \in \mathbb{C}^n$, see e.g. [197]. The $3K$-complex-dimensional Gaußian integral in (4.90) is brought into this form by defining

$$\mathtt{u} = \left(\begin{array}{c} \mathrm{Re}\,\boldsymbol{w}_\alpha \\ \hline \mathrm{Im}\,\boldsymbol{w}_\alpha \end{array}\right), \quad \mathcal{A} = -\mathcal{E}^t\check{\mathtt{H}}\mathcal{E}\,, \quad \mathcal{E} = \left(\begin{array}{c|c} 1_{3K} & i1_{3K} \\ \hline 1_{3K} & -i1_{3K} \end{array}\right), \quad \mathtt{b} = \sqrt{2}\left(\begin{array}{c} \boldsymbol{\sigma}^{\mathrm{o}}_\alpha \\ \hline \overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha \\ \hline \overline{\boldsymbol{\sigma}}^{\mathrm{o}}_\alpha \\ \hline -i\boldsymbol{\sigma}^{\mathrm{o}}_\alpha \\ \hline -i\overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha \\ \hline -i\overline{\boldsymbol{\sigma}}^{\mathrm{o}}_\alpha \end{array}\right). \tag{4.93}$$

One easily verifies that this matrix $\mathcal{A}$ is symmetric and all its eigenvalues are equal to 2. Consequently (4.92) can be applied and we obtain

$$\Phi_\alpha(\boldsymbol{\sigma}_\alpha, \overline{\boldsymbol{\sigma}}_\alpha) = e^{-\overline{\boldsymbol{\sigma}}^t_\alpha\boldsymbol{\sigma}_\alpha}\delta_{\mathbb{C}^K}(\overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha - \mathcal{C}\boldsymbol{\sigma}^{\mathrm{o}}_\alpha) e^{(\mathcal{C}^\dagger\overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha)^t\overline{\boldsymbol{\sigma}}^{\mathrm{o}}_\alpha + (\overline{\boldsymbol{\sigma}}^{\mathrm{o}}_\alpha)^t\mathcal{C}^\dagger\overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha} = \delta_{\mathbb{C}^K}(\overline{\boldsymbol{\sigma}}^{\mathrm{d}}_\alpha - \mathcal{C}\boldsymbol{\sigma}^{\mathrm{o}}_\alpha)\,, \tag{4.94}$$

where we made use of the delta function for the last equality. Finally, to arrive at the desired result (4.81), we rename the variables σ^i_α contained in $\boldsymbol{\sigma}_\alpha$ according to (4.70) into λ^i_α. Q.E.D.

4.2.5 Extension to Superalgebras

In the previous sections we detailed the change of basis from oscillator to spinor helicity-like variables for the bosonic algebra $\mathfrak{u}(p,p)$. In particular, we explained how the symmetry generators and the integrand of the unitary Graßmannian matrix model (4.24) are transformed. To cover the case relevant for tree-level $\mathcal{N}=4$ SYM scattering amplitudes, we have to extend these calculations to superalgebras. In the present section we show how our results generalize from $\mathfrak{u}(p,p)$ to the superalgebra $\mathfrak{u}(p,p|r+s)$.

Let us start by discussing the realization of fermionic oscillators is terms of a Graßmann algebra, cf. [201, 197, 196]. This can be viewed as the fermionic analogue of the Bargmann representation from section 4.2.1. Consider a family of fermionic oscillators on a Fock space,

$$\{\mathbf{c}_a,\bar{\mathbf{c}}_b\}=\delta_{ab}\,,\quad {\mathbf{c}_a}^\dagger=\bar{\mathbf{c}}_a\,,\quad \mathbf{c}_a|0\rangle=0\,, \tag{4.95}$$

where $a,b=1\,\ldots,r$ and the bracket denotes the anticommutator. These oscillators can be thought of as being associated with the $\mathfrak{u}(0|r)$ subalgebra of $\mathfrak{u}(p,p|r+s)$ according to (2.32). However, this interpretations is not yet of importance in this paragraph. The commutation relations and the action on the vacuum in (4.95) are realized using a Graßmann algebra with anticommuting generators χ_a,

$$\bar{\mathbf{c}}\mapsto\chi\,,\quad \mathbf{c}\mapsto\partial_\chi\,,\quad |0\rangle\mapsto 1\,, \tag{4.96}$$

where $\bar{\mathbf{c}}=(\bar{\mathbf{c}}_a)$ and $\chi=(\chi_a)$ are r-component column vectors. In order to implement the adjoint in (4.95) we need additional structure. We define the one-dimensional Berezin integral as $\int \mathrm{d}\chi_a(\alpha+\beta\chi_a)=\int(\alpha-\beta\chi_a)\,\mathrm{d}\chi_a=\beta$ for complex numbers α,β. Multi-dimensional integrals are obtained by iteration. Note that χ_a and $\mathrm{d}\chi_b$ anticommute. Furthermore, we append the r anticommuting generators $\overline{\chi}_a$ to the original Graßmann algebra and demand $\{\overline{\chi}_a,\chi_b\}=0$. This extended Graßmann algebra is equipped with an antilinear antiinvolution that we denote, with some abuse of notation, also by $\bar{\cdot}$. It is defined by mapping χ_a to the generator $\overline{\chi}_a$ of the extended algebra. Furthermore, we have $\overline{\chi_a\chi_b}=\overline{\chi}_b\overline{\chi}_a$ and $\overline{\overline{\chi}}_a=\chi_a$. These structures allow us to define the inner product

$$\langle\Psi(\chi),\Phi(\chi)\rangle=\int \mathrm{d}^r\chi\,\mathrm{d}^r\overline{\chi}e^{-\chi^t\overline{\chi}}\,\overline{\Psi(\chi)}\Phi(\chi)\,, \tag{4.97}$$

where $\mathrm{d}^r\chi=\mathrm{d}\chi_1\cdots\mathrm{d}\chi_r$. Here $\Psi(\chi)$ and $\Phi(\chi)$ are "holomorphic" in the sense that they do not depend on the generators $\overline{\chi}_a$. One then verifies that with respect to this inner product ${\chi_a}^\dagger=\partial_{\chi_a}$, i.e. (4.97) implements the adjoint in (4.95). The realization of the fermionic oscillators in (4.96) and the inner product (4.97) are very much reminiscent, respectively, of (4.53) and (4.54) from the Bargmann representation of bosonic oscillators.

We turn to the superoscillators that are used to build the representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_c$ of $\mathfrak{u}(p,p|r+s)$ in (2.34) and (2.36), respectively. We want to reformulate the generators of these representations in such a way that they can be identified with those appearing for $\mathcal{N}=4$ SYM amplitudes in (1.30). To this end, the bosonic oscillators among the superoscillators are realized using the Bargmann representation of section 4.2.1. For the fermionic ones we employ the realization in terms of a Graßmann algebra as just explained. Let us state our naming of the variables and indices based on the creation operators, cf.

(2.32),

$$(\bar{\mathbf{A}}_{\mathcal{A}}) = \left(\begin{array}{c} \bar{\mathbf{a}}_\alpha \\ \hdashline \bar{\mathbf{c}}_a \\ \hdashline \bar{\mathbf{b}}_{\dot\alpha} \\ \hdashline \bar{\mathbf{d}}_{\dot a} \end{array}\right) \mapsto \left(\begin{array}{c} z_\alpha \\ \hdashline \chi_a \\ \hdashline z_{\alpha+p} \\ \hdashline \psi_{\dot a} \end{array}\right) \tag{4.98}$$

with complex commuting variables z_α and $z_{\alpha+p}$ as well as anticommuting Graßmann variables χ_a and $\psi_{\dot a}$. The index ranges of the bosonic variables are $\alpha, \dot\alpha = 1, \dots, p$ and those of the fermionic ones read $a = 1, \dots, r$ and $\dot a = 1, \dots, s$. Thus $\mathcal{A} = 1, \dots, 2p + r + s$. We already dealt with the bosonic variables in the previous sections, where we expressed them in terms of the spinor helicity-like variables λ_α. Therefore we can concentrate on the fermionic ones from now on. As in the bosonic case, cf. (4.70), there is a slight distinction between ordinary sites with representations $\mathcal{D}_c$ and dual ones with $\bar{\mathcal{D}}_c$. At the ordinary sites we just rename

$$\chi \mapsto \theta\,, \quad \partial_\chi \mapsto \partial_\theta\,, \quad \psi \mapsto \eta\,, \quad \partial_\psi \mapsto \partial_\eta\,, \quad 1 \mapsto 1\,. \tag{4.99}$$

At the dual sites we apply a fermionic analogue of the Fourier transformation,

$$\Phi(\chi, \psi) \mapsto \Phi(\theta, \eta) = \int e^{\theta^t \chi - \eta^t \psi}\, \Phi(\chi, \psi)\, \mathrm{d}^r\chi\, \mathrm{d}^s\psi\,. \tag{4.100}$$

This transformation amounts to

$$\chi \mapsto \partial_\theta\,, \quad \partial_\chi \mapsto \theta\,, \quad \psi \mapsto -\partial_\eta\,, \quad \partial_\psi \mapsto -\eta\,, \quad 1 \mapsto (-1)^r \theta_r \cdots \theta_1 \eta_1 \cdots \eta_s\,. \tag{4.101}$$

The "measure" in (4.100) is on the right hand side of the integrand in order to avoid additional signs in (4.101). Let us apply these transformations to the fermionic variables of the generators $\mathbf{J}_{\mathcal{AB}}$ of $\mathcal{D}_c$ and $\bar{\mathbf{J}}_{\mathcal{AB}}$ of $\bar{\mathcal{D}}_c$ in the $\mathfrak{u}(p,p|r+s)$ case, which are given in terms of oscillators in (2.34) and (2.36), respectively. Together with the result (4.71) for the bosonic variables this yields

$$\begin{aligned}
(\mathbf{J}_{\mathcal{AB}}) \leftrightarrow (\mathfrak{J}_{\mathcal{AB}}) &= D \left(\begin{array}{c:c:c:c} \lambda_\alpha \partial_{\lambda_\beta} & \lambda_\alpha \partial_{\theta_b} & \lambda_\alpha \overline{\lambda}_\beta & \lambda_\alpha \eta_{\dot b} \\ \hdashline \theta_a \partial_{\lambda_\beta} & \theta_a \partial_{\theta_b} & \theta_a \overline{\lambda}_\beta & \theta_a \eta_{\dot b} \\ \hdashline -\partial_{\overline{\lambda}_\alpha} \partial_{\lambda_\beta} & -\partial_{\overline{\lambda}_\alpha} \partial_{\theta_b} & -\partial_{\overline{\lambda}_\alpha} \overline{\lambda}_\beta & -\partial_{\overline{\lambda}_\alpha} \eta_{\dot b} \\ \hdashline \partial_{\eta_{\dot a}} \partial_{\lambda_\beta} & \partial_{\eta_{\dot a}} \partial_{\theta_b} & \partial_{\eta_{\dot a}} \overline{\lambda}_\beta & \partial_{\eta_{\dot a}} \eta_{\dot b} \end{array}\right) D^{-1}\,, \\
(\bar{\mathbf{J}}_{\mathcal{AB}}) \leftrightarrow (\bar{\mathfrak{J}}_{\mathcal{AB}}) &= D \left(\begin{array}{c:c:c:c} \partial_{\lambda_\beta} \lambda_\alpha & \partial_{\theta_b} \lambda_\alpha & -\overline{\lambda}_\beta \lambda_\alpha & \eta_{\dot b} \lambda_\alpha \\ \hdashline \partial_{\lambda_\beta} \theta_a & -\partial_{\theta_b} \theta_a & -\overline{\lambda}_\beta \theta_a & -\eta_{\dot b} \theta_a \\ \hdashline \partial_{\lambda_\beta} \partial_{\overline{\lambda}_\alpha} & \partial_{\theta_b} \partial_{\overline{\lambda}_\alpha} & -\overline{\lambda}_\beta \partial_{\overline{\lambda}_\alpha} & \eta_{\dot b} \partial_{\overline{\lambda}_\alpha} \\ \hdashline \partial_{\lambda_\beta} \partial_{\eta_{\dot a}} & -\partial_{\theta_b} \partial_{\eta_{\dot a}} & -\overline{\lambda}_\beta \partial_{\eta_{\dot a}} & -\eta_{\dot b} \partial_{\eta_{\dot a}} \end{array}\right) D^{-1}
\end{aligned} \tag{4.102}$$

with the $(p+p|r+s) \times (p+p|r+s)$ supermatrix

$$D = \left(\begin{array}{c:c:c:c} 1_p & 0 & 1_p & 0 \\ \hdashline 0 & \sqrt{2}\, 1_r & 0 & 0 \\ \hdashline -1_p & 0 & 1_p & 0 \\ \hdashline 0 & 0 & 0 & \sqrt{2}\, 1_s \end{array}\right). \tag{4.103}$$

This matrix is even with respect to the grading (2.1) and thus satisfies the criteria of section 4.2.3. Therefore the similarity transformation in (4.102) can be absorbed in a redefinition of the Yangian generators. Furthermore, we remark that the ordinary and dual generators in (4.102) are identical up to some signs and a shift,

$$\bar{\mathfrak{J}}_{\mathcal{A}\mathcal{B}}\Big|_{(\lambda,\overline{\lambda})\mapsto(\lambda,-\overline{\lambda})} = \mathfrak{J}_{\mathcal{A}\mathcal{B}} + \delta_{\mathcal{A}\mathcal{B}}(-1)^{|\mathcal{A}|}\,. \tag{4.104}$$

This generalizes the bosonic relation (4.73). We also translate the central elements (2.35) and (2.37), which function as representation labels, into the new basis,

$$\begin{aligned} \mathbf{C} = \operatorname{tr}(\mathbf{J}_{\mathcal{A}\mathcal{B}}) \leftrightarrow \mathfrak{C} &= \sum_{\alpha=1}^{p}(\lambda_\alpha\partial_{\lambda_\alpha} - \overline{\lambda}_\alpha\partial_{\overline{\lambda}_\alpha}) + \sum_{a=1}^{r}\theta_a\partial_{\theta_a} - \sum_{\dot{a}=1}^{s}\eta_{\dot{a}}\partial_{\eta_{\dot{a}}} - p + s\,,\\ \bar{\mathbf{C}} = \operatorname{tr}(\bar{\mathbf{J}}_{\mathcal{A}\mathcal{B}}) \leftrightarrow \bar{\mathfrak{C}} &= \sum_{\alpha=1}^{p}(\lambda_\alpha\partial_{\lambda_\alpha} - \overline{\lambda}_\alpha\partial_{\overline{\lambda}_\alpha}) + \sum_{a=1}^{r}\theta_a\partial_{\theta_a} - \sum_{\dot{a}=1}^{s}\eta_{\dot{a}}\partial_{\eta_{\dot{a}}} + p - r\,. \end{aligned} \tag{4.105}$$

Let us for the moment concentrate on the algebra $\mathfrak{u}(2,2|r+s=0+4)$, which appears in our discussion of the superconformal symmetry of $\mathcal{N}=4$ SYM amplitudes in section 1.3.4. In this case we can identify the generators $\mathfrak{J}_{\mathcal{A}\mathcal{B}}$ of $\mathcal{D}_c$ in (4.102) with those in (1.30) for $\tilde{\lambda} = +\overline{\lambda}$, where we neglect the similarity transformation with the matrix D. The generators $\bar{\mathfrak{J}}_{\mathcal{A}\mathcal{B}}$ of $\bar{\mathcal{D}}_c$ in (4.102) match those in (1.30) for $\tilde{\lambda} = -\overline{\lambda}$ up to the same similarity transformation and a shift as in (4.104). Furthermore, for this algebra the expressions for $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ in (4.105) coincide because $-p+s = p-r = 2$. The condition that the eigenvalue of these central elements equals $c=0$ is identical to the constraint on the "superhelicity" of the amplitudes in (1.21). Thus in our language the tree-level amplitudes of $\mathcal{N}=4$ SYM transform in the representations $\mathcal{D}_0$ and $\bar{\mathcal{D}}_0$ of $\mathfrak{u}(2,2|0+4)$. These belong, respectively, to particles with positive and negative energy as we know from the last paragraphs of section 4.2.2.

Finally, we want to apply the change of basis to the integrand of the Graßmannian matrix model (4.24). We observe that this integrand factorizes into one part containing the bosonic oscillators and one with the fermionic ones,

$$|\Phi\rangle = e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ\mathcal{C}^\dagger)}|0\rangle = e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_{\bullet\mathrm{b}}^t + \mathbf{I}_{\circ\mathrm{b}}\mathcal{C}^\dagger)}|0\rangle_{\mathrm{b}}\, e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_{\bullet\mathrm{f}}^t + \mathbf{I}_{\circ\mathrm{f}}\mathcal{C}^\dagger)}|0\rangle_{\mathrm{f}} = |\Phi\rangle_{\mathrm{b}}|\Phi\rangle_{\mathrm{f}}\,. \tag{4.106}$$

This factorization is based on the structure of the entries (4.3) of $\mathbf{I}_{\substack{\bullet\\\circ}}$, that can be written as $(k \overset{\bullet}{\circ} l) = (k \overset{\bullet}{\circ} l)_{\mathrm{b}} + (k \overset{\bullet}{\circ} l)_{\mathrm{f}}$. The subscripts b and f refer to the parts with bosonic and fermionic oscillators, respectively. We already studied the transformation of $|\Phi\rangle_{\mathrm{b}}$ to spinor helicity-like variables in section 4.2.4. Thus we can concentrate on $|\Phi\rangle_{\mathrm{f}}$ here. We realize the fermionic oscillators as in (4.98). Then we apply the replacement (4.99) at the ordinary sites and the Fourier transformation (4.100) at the dual ones. In our conventions the "measure" in the Fourier transformation from site k is left of that from site $k+1$. This yields

$$|\Phi\rangle_{\mathrm{f}} \mapsto \Phi(\boldsymbol{\theta},\boldsymbol{\eta})_{\mathrm{f}} = \epsilon\,\delta^{0|rK}(\boldsymbol{\theta}^{\mathrm{d}} + \mathcal{C}\boldsymbol{\theta}^{\mathrm{o}})\delta^{0|sK}(\boldsymbol{\eta}^{\mathrm{d}} - \overline{\mathcal{C}}\boldsymbol{\eta}^{\mathrm{o}}) \tag{4.107}$$

with the sign $\epsilon = (-1)^{\frac{1}{2}rs(K-1)K+rK}$. Here we arranged the Graßmann variables into the matrices

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}^{\mathrm{d}} \\ \hdashline \boldsymbol{\eta}^{\mathrm{o}} \end{pmatrix}, \quad \boldsymbol{\eta}^{\mathrm{d}} = \begin{pmatrix} \eta_1^1 & \cdots & \eta_s^1 \\ \vdots & & \vdots \\ \eta_1^K & \cdots & \eta_s^K \end{pmatrix}, \quad \boldsymbol{\eta}^{\mathrm{o}} = \begin{pmatrix} \eta_1^{K+1} & \cdots & \eta_s^{K+1} \\ \vdots & & \vdots \\ \eta_1^{2K} & \cdots & \eta_s^{2K} \end{pmatrix},$$
$$\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\theta}^{\mathrm{d}} \\ \hdashline \boldsymbol{\theta}^{\mathrm{o}} \end{pmatrix}, \quad \boldsymbol{\theta}^{\mathrm{d}} = \begin{pmatrix} \theta_1^1 & \cdots & \theta_r^1 \\ \vdots & & \vdots \\ \theta_1^K & \cdots & \theta_r^K \end{pmatrix}, \quad \boldsymbol{\theta}^{\mathrm{o}} = \begin{pmatrix} \theta_1^{K+1} & \cdots & \theta_r^{K+1} \\ \vdots & & \vdots \\ \theta_1^{2K} & \cdots & \theta_r^{2K} \end{pmatrix}. \tag{4.108}$$

The fermionic delta functions occurring in (4.107) are defined by

$$\delta^{0|uv}(\mathtt{A}) = \prod_{i=1}^{u} \prod_{j=1}^{v} \mathtt{A}_{ij} \tag{4.109}$$

for a Graßmann-valued $u \times v$ matrix $\mathtt{A} = (\mathtt{A}_{ij})$. Factors with smaller values of the indices appear left in the products. Notice that with a bosonic $u \times u$ matrix $\mathtt{B}$ we have $\delta^{0|uv}(\mathtt{BA}) = \det(\mathtt{B})^v \delta^{0|uv}(\mathtt{A})$. We conclude by combining the transformation of $|\Phi\rangle_{\mathrm{f}}$ from (4.107) with that of the bosonic part $|\Phi\rangle_{\mathrm{b}}$ in (4.81),

$$|\Phi\rangle \mapsto \Phi(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}, \boldsymbol{\theta}, \boldsymbol{\eta}) = \epsilon\, \delta_{\mathbb{C}}^{pK|0}(\boldsymbol{\lambda}^{\mathrm{d}} + \mathcal{C}\boldsymbol{\lambda}^{\mathrm{o}}) \delta^{0|rK}(\boldsymbol{\theta}^{\mathrm{d}} + \mathcal{C}\boldsymbol{\theta}^{\mathrm{o}}) \delta^{0|sK}(\boldsymbol{\eta}^{\mathrm{d}} - \overline{\mathcal{C}}\boldsymbol{\eta}^{\mathrm{o}}). \tag{4.110}$$

This is the integrand of the unitary Graßmannian matrix model (4.24) for oscillator representations of $\mathfrak{u}(p,p|m)$ expressed in analogues of the spinor helicity variables for $\mathcal{N} = 4$ SYM scattering amplitudes.

4.3 Graßmannian Integral in Spinor Helicity Variables

Here we utilize the change of basis from oscillators to spinor helicity-like variables, which we just derived, to obtain a unitary Graßmannian integral formula in terms of the latter variables. This formula is explained in section 4.3.1. There we also point out its differences to the original Graßmannian integral proposal, which we reviewed in section 1.3.5. What is more, we find a tight relation between the unitary of the integration contour and momentum conservation, which is the topic of section 4.3.2. Section 4.3.3 is devoted to the evaluation of the integral for sample Yangian invariants. In particular, we study examples that are related to certain tree-level gluon amplitudes, superamplitudes of $\mathcal{N} = 4$ SYM and to integrable deformations thereof.

4.3.1 Unitary Graßmannian Integral

After we transformed the integrand of the unitary Graßmannian matrix model (4.24) in the foregoing section 4.2 and summarized the result in (4.110), we are able to state the transformation of the entire integral. This results in a refined *Graßmannian integral in spinor helicity-like variables.* It computes Yangian invariants for representations of $\mathfrak{u}(p,p|r+s)$ with $N = 2K$ sites, out of which the first K are dual, and reads

$$\Psi_{2K,K}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}, \boldsymbol{\theta}, \boldsymbol{\eta}) = \epsilon \chi_K^{-1} \int_{U(K)} [\mathrm{d}\mathcal{C}] \mathscr{F}(\mathcal{C})\, \delta_{\mathbb{C}}^{pK|0}(C\boldsymbol{\lambda}) \delta^{0|rK}(C\boldsymbol{\theta}) \delta^{0|sK}(C^{\perp}\boldsymbol{\eta}). \tag{4.111}$$

In this formula the prefactor involves the sign ϵ introduced after (4.107). The $U(K)$ invariant Haar measure $[\mathrm{d}\mathcal{C}]$ is defined in (4.35). Its normalization is denoted by χ_K. The integrand $\mathscr{F}(\mathcal{C})$ is specified in (4.30). Recall also its explicit form for $N = 2, 4, 6$ in (4.31), (4.32) and (4.33), respectively. The unitary $K \times K$ matrix $\mathcal{C}$ is embedded into the $K \times 2K$ matrix $C = (\ 1_K \ \vdots\ \mathcal{C}\ )$, which is an element of the Graßmannian $\mathrm{Gr}(2K, K)$, cf. (1.41). We also introduced $C^\perp = (\ 1_K \ \vdots\ -\overline{\mathcal{C}}\ )$ satisfying $C(C^\perp)^t = 0$. The external bosonic and fermionic variables are encoded in the matrices

$$\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}^{\mathrm{d}} \\ \hdashline \boldsymbol{\lambda}^{\mathrm{o}} \end{pmatrix}, \quad \boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}^{\mathrm{d}} \\ \hdashline \boldsymbol{\eta}^{\mathrm{o}} \end{pmatrix}, \quad \boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\theta}^{\mathrm{d}} \\ \hdashline \boldsymbol{\theta}^{\mathrm{o}} \end{pmatrix}. \tag{4.112}$$

Here the $K \times p$ blocks $\boldsymbol{\lambda}^{\mathrm{d}} = (\lambda^k_\alpha)$ and $\boldsymbol{\lambda}^{\mathrm{o}} = (\lambda^l_\alpha)$ contain, respectively, the spinor helicity-like variables at the dual sites $k = 1, \ldots, K$ and ordinary sites $l = K+1, \ldots, 2K$, see (4.82). Analogously, the fermionic variables are arranged into the $K \times s$ blocks $\boldsymbol{\eta}^{\mathrm{d}} = (\eta^k_{\dot{a}})$, $\boldsymbol{\eta}^{\mathrm{o}} = (\eta^l_{\dot{a}})$ and into the $K \times r$ blocks $\boldsymbol{\theta}^{\mathrm{d}} = (\theta^k_a)$, $\boldsymbol{\theta}^{\mathrm{o}} = (\theta^l_a)$, see (4.108). Lastly, the complex bosonic delta function in (4.111) is defined as the delta function of the real part of its the argument times that of the imaginary part, see (4.91). The definition of the fermionic delta function is provided in (4.109).

Let us compare our refined formula (4.111) to the original proposal of the Graßmannian integral for tree-level $\mathcal{N} = 4$ SYM amplitudes, as reviewed in the introductory section 1.3.5. We know from the identification of the generators belonging to $\mathfrak{u}(p, p|r + s)$ representations with those of the amplitudes in section 4.2.5 that we have to restrict to the algebra $\mathfrak{u}(2, 2|0+4)$ for this comparison. This is essentially the superconformal algebra discussed in section 1.3.4. First, we observe some notational differences. The matrix $C^\perp$ is determined by $C(C^\perp)^t = 0$ only up to a $GL(\mathbb{C}^K)$ transformation. Our choice of this matrix after (4.111) differs from that in (1.42) by such a transformation. Moreover, the roles of the matrices C and $C^\perp$ in (4.111) and the original formula (1.43) are exchanged. Both matrices are elements of the same Graßmannian $\mathrm{Gr}(2K, K)$ in the case $N = 2K$ under consideration. Therefore this exchange may be viewed as an alternative choice of parameterization.

Of considerably more importance are conceptual differences. These address the issues of the original Graßmannian integral that we identified in section 1.3.5. In (4.111) we work at all times in real Minkowski space with $(1, 3)$ signature. That is, the complex spinor helicity variables $\lambda^i = (\lambda^i_\alpha)$ and $\tilde{\lambda}^i = (\tilde{\lambda}^i_{\dot{\alpha}})$, which according to (1.4) make up the particle momenta p^i, obey the reality condition (1.5), see the discussion in section 4.2.5. We have

$$\tilde{\lambda}^i = \begin{cases} -\overline{\lambda}^i & \text{for dual sites} \qquad i = 1, \ldots, K\,, \\ \overline{\lambda}^i & \text{for ordinary sites} \qquad i = K+1, \ldots, 2K\,. \end{cases} \tag{4.113}$$

These reality conditions are imperative considering the oscillator representations we started out with in section 4.1. In contrast, in the original proposal (1.43) one works in $(2, 2)$ signature or a complexified momentum space, which entails, respectively, real or complex *independent* variables $\lambda^i, \tilde{\lambda}^i$. Furthermore, our formula (4.111) does not feature a delta function involving the matrix $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}^i_{\dot{\alpha}})$ as in (1.43). In our setting the reality conditions (4.113) yield

$$\tilde{\boldsymbol{\lambda}} = \begin{pmatrix} \tilde{\boldsymbol{\lambda}}^{\mathrm{d}} \\ \hdashline \tilde{\boldsymbol{\lambda}}^{\mathrm{o}} \end{pmatrix} = \begin{pmatrix} -\overline{\boldsymbol{\lambda}}^{\mathrm{d}} \\ \hdashline \overline{\boldsymbol{\lambda}}^{\mathrm{o}} \end{pmatrix}. \tag{4.114}$$

Thus $\tilde{\boldsymbol{\lambda}}$ is determined by $\boldsymbol{\lambda}$ and does not have to be constrained by a separate delta function. Let us also stress that the complex bosonic delta function $\delta_{\mathbb{C}}$ in (4.111) is defined by (4.91) in terms of ordinary real delta functions and thus differs from the somewhat formal δ_* in

(1.43). This brings us to another issue raised in the context of the original Graßmannian integral (1.43). In that approach a number of delta functions is usually eliminated in a purely algebraic fashion without the specification of a corresponding contour of integration. Then a contour is enforced "by hand" onto the remaining integral. The unitary contour in (4.111) is supposed to unify both steps in a natural way. As we shall see in the next section, the unitary of the integration variable $\mathcal{C}$ and the reality conditions on the spinor helicity variables are in fact tightly interlocked.

On a different note, we reviewed the extension of the Graßmannian integral (1.43) to deformed amplitudes in section 1.3.6. In the resulting formula (1.54) the challenge of selecting an appropriate contour of integration seems to be quite involved due to the branch cuts of the integrand. In particular, a satisfactory solution is not known even for the six-particle NMHV amplitude. The situation changes drastically for our refined Graßmannian integral in (4.111). Its integrand $\mathscr{F}(\mathcal{C})$ defined in (4.30) also incorporates deformation parameters. Nevertheless, it is manifestly free from any branch cuts as discussed in section 4.1.4.1. Recall that the derivation of (4.30) from the standard form of the integrand in (4.25) is heavily based on the unitarity of $\mathcal{C}$. Furthermore, it makes decisive use of integer representation labels c_i, as opposed to complex ones employed in the original deformed Graßmannian integral (1.54).

Despite these numerous advantages of the Graßmannian integral (4.111) with a unitary contour, it remains to be shown whether it reproduces the well-known expressions for the $\mathcal{N}=4$ SYM tree-level amplitudes. This will be addressed in section 4.3.3 by evaluating (4.111) for sample invariants.

4.3.2 Momentum Conservation and Unitarity of Contour

Before we investigate sample invariants, it is instructive to study the relation between momentum conservation and the unitary contour in the Graßmannian integral (4.111) on a general level. As this analysis applies to the $\mathfrak{u}(p,p|r+s)$ case of the integral, we work with an appropriate generalization of four-dimensional Minkowski momenta for this algebra.

To begin with, we have to introduce some notation. We define the "momentum" and two notions of "supermomenta" by

$$P_{\alpha\dot{\alpha}}=\sum_{i=1}^{N}\lambda^i_\alpha\tilde{\lambda}^i_{\dot{\alpha}}\,,\quad Q_{\alpha\dot{a}}=\sum_{i=1}^{N}\lambda^i_\alpha\eta^i_{\dot{a}}\,,\quad \hat{Q}_{\dot{\alpha}a}=\sum_{i=1}^{N}\tilde{\lambda}^i_{\dot{\alpha}}\theta^i_a \tag{4.115}$$

with the index ranges $\alpha,\dot{\alpha}=1,\ldots,p$, $a=1,\ldots,r$ and $\dot{a}=1,\ldots,s$. For the algebra $\mathfrak{u}(2,2|0+4)$ these reduce to the four-dimensional Minkowski momentum $P_{\alpha\dot{\alpha}}$ and the supermomentum $Q_{\alpha\dot{a}}$ defined in (1.16) and (1.24), respectively. The $p\times p$ matrix $P=(P_{\alpha\dot{\alpha}})$ is Hermitian because of (4.113). This observation allows us to specify the p^2-dimensional real bosonic delta function

$$\delta^{p^2|0}(P)=\prod_{\alpha=\dot{\alpha}=1}^{p}\delta(P_{\alpha\dot{\alpha}})\prod_{\substack{\alpha,\dot{\alpha}=1\\ \alpha>\dot{\alpha}}}^{p}\delta(\operatorname{Re}P_{\alpha\dot{\alpha}})\delta(\operatorname{Im}P_{\alpha\dot{\alpha}})\,. \tag{4.116}$$

It will be referred to as "momentum conserving" delta function from now on. Furthermore, let us introduce the $p\times s$ matrix $Q=(Q_{\alpha\dot{a}})$ and the $p\times r$ matrix $\hat{Q}=(\hat{Q}_{\dot{\alpha}a})$ with Graßmann-valued entries. The "supermomentum conserving" delta functions are

$$\delta^{0|ps}(Q)=\prod_{\alpha=1}^{p}\prod_{\dot{a}=1}^{r}Q_{\alpha\dot{a}}\,,\quad \delta^{0|pr}(\hat{Q})=\prod_{\dot{\alpha}=1}^{p}\prod_{a=1}^{s}\hat{Q}_{\dot{\alpha}a}\,. \tag{4.117}$$

Recall (4.109) for the definition of a fermionic delta function.

Next, we show that the Graßmannian integral (4.111) with the unitary contour implies momentum conservation. From the delta functions in the Graßmannian integral (4.111) we obtain

$$C\boldsymbol{\lambda} = 0\,, \quad C^{\perp}\tilde{\boldsymbol{\lambda}} = 0\,, \quad C\boldsymbol{\theta} = 0\,, \quad C^{\perp}\boldsymbol{\eta} = 0\,. \tag{4.118}$$

The second equation is minus the complex conjugate of the first one. Nevertheless, it is instructional to display it here explicitly. Of course, the first two equations are only valid on the support of the bosonic delta function in (4.111). Analogously, the remaining equations hold in the presence of the fermionic delta functions in (4.111). The equations in (4.118) together with, importantly, the unitarity of the integration variable $\mathcal{C}$ imply momentum and supermomentum conservation,

$$\begin{aligned} \boldsymbol{\lambda}^t\tilde{\boldsymbol{\lambda}} = 0 \quad &\Leftrightarrow \quad P_{\alpha\dot{\alpha}} = 0\,,\\ \boldsymbol{\lambda}^t\boldsymbol{\eta} = 0 \quad &\Leftrightarrow \quad Q_{\alpha\dot{a}} = 0\,,\\ \tilde{\boldsymbol{\lambda}}^t\boldsymbol{\theta} = 0 \quad &\Leftrightarrow \quad \hat{Q}_{\dot{\alpha}a} = 0\,. \end{aligned} \tag{4.119}$$

This is easily verified after splitting the matrices $\boldsymbol{\lambda}, \tilde{\boldsymbol{\lambda}}, \boldsymbol{\theta}, \boldsymbol{\eta}$ into "dual" and "ordinary" blocks as in (4.112) and (4.114). Therefore the Yangian invariant $\Psi_{2K,K}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}, \boldsymbol{\theta}, \boldsymbol{\eta})$ computed by the Graßmannian integral (4.111) is proportional to the momentum and supermomentum conserving delta functions (4.116) and (4.117), respectively.

We move on to show, in a certain sense, the converse statement, i.e. that demanding momentum conservation implies the unitary of the integration variable $\mathcal{C}$ in (4.111). Let us first state our assumptions. We do not specify a contour of integration in (4.111) and hence at the outset $\mathcal{C} \in GL(\mathbb{C}^K)$. The spinor helicity-like variables $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$ satisfy the reality conditions (4.114). The bosonic delta function in the integral (4.111) enforces $C\boldsymbol{\lambda} = 0$. Lastly, we assume that the integration contour, and therefore $\mathcal{C}$, does not depend on the external data $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$. Under these premises, momentum conservation becomes

$$0 = \boldsymbol{\lambda}^t\tilde{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}^{\circ})^t(1_K - \mathcal{C}^t\overline{\mathcal{C}})\overline{\boldsymbol{\lambda}}^{\circ}\,, \tag{4.120}$$

which has to hold for any complex $K \times p$ matrix $\boldsymbol{\lambda}^{\circ}$. We rephrase this equation by introducing $\mathtt{h} = \overline{\boldsymbol{\lambda}}^{\circ}$ and $\mathtt{A} = \mathtt{A}^{\dagger} = 1_K - \mathcal{C}^t\overline{\mathcal{C}}$. This yields

$$0 = \mathtt{h}^{\dagger}\mathtt{A}\mathtt{h}\,, \tag{4.121}$$

which has to be satisfied for all $\mathtt{h}$. Diagonalizing the Hermitian matrix $\mathtt{A}$ by means of a unitary transformation, we see that this equation implies $\mathtt{A} = 0$. Thus $\mathcal{C}$ has to be unitary. In this spirit momentum conservation implies a unitary contour.

We should add a comment regarding one of our assumptions in this argument. In the usual Graßmannian integral approach to $\mathcal{N} = 4$ SYM amplitudes, which we reviewed in section 1.3.5, the integration contour does depend on the external data $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$. It encircles certain poles of the integrand in (1.43). The positions of these poles depend on $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$, and therefore so does the contour. This violates our assumption. Nevertheless, in this thesis we retain the assumption of a contour that is independent of the external data. In fact, it is very natural from the point of view put forward in this thesis. We not only have the Graßmannian integral (4.111) in spinor helicity variables but also know how to transform it into the oscillator form (4.24). In this basis a contour which depends on the "values" of the oscillators does not seem to be well-defined. In contrast, the unitary contour is known to yield correct oscillator sample Yangian invariants, see section 4.1.5. Moreover, we proved the Yangian invariance of the Graßmannian integral (4.24) for this contour. Let us emphasize that this proof also applies to the spinor helicity version (4.111) of the integral.

4.3.3 Sample Invariants and Amplitudes

Now we are in a position to actually evaluate the unitary Graßmannian integral (4.111) in order to obtain sample Yangian invariants in spinor helicity-like variables. We focus on invariants with representations of the algebra $\mathfrak{u}(2,2)$ and the superalgebra $\mathfrak{u}(2,2|0+4)$ because of their relevance for gluon amplitudes and superamplitudes of $\mathcal{N}=4$ SYM, respectively. We identify the four-site invariant with the four-particle MHV amplitude. Furthermore, the six-site invariant is computed and its relation to the six-particle NMHV amplitude is discussed. Recall that this is the first case where one has to impose a contour "by hand" in the usual Graßmannian integral approach of section 1.3.5. More generally, our sample invariants contain deformation parameters, which allow us to relate them to the deformed amplitudes of section 1.3.6. Before we turn our attention to the above-mentioned algebras, it is instructive to compute some sample invariants for $\mathfrak{u}(1,1)$. These share key features with the higher rank examples but are technically easier to compute. Even before that, we discuss some tools which will be of great utility for the evaluation of the unitary Graßmannian integral (4.111) in all the examples considered. Let us also mention that some additional sample invariants are computed in appendix B.3.

4.3.3.1 Tools for Evaluation of Integrals

The unitary Graßmannian integral (4.111) contains a complex bosonic delta function that we have to manipulate in the course of evaluating the integral. Thus we recall some of its properties. We begin with the definition in terms of real delta functions. For those we have

$$\int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\,\delta(x)\delta(y)f(x,y) = f(0,0) \tag{4.122}$$

for a suitable test function $f(x,y)$. Let us introduce the complex coordinate $z = x + iy$. Then the measure reads $(2i)^{-1}\,\mathrm{d}\overline{z}\,\mathrm{d}z = \mathrm{d}x\,\mathrm{d}y$. Defining $\delta_{\mathbb{C}}(z) = \delta(x)\delta(y)$ and denoting the test function by $g(z,\overline{z}) = f(x,y)$, the above equation turns into

$$\int_{\mathbb{C}} \frac{\mathrm{d}\overline{z}\,\mathrm{d}z}{2i}\delta_{\mathbb{C}}(z)g(z,\overline{z}) = g(0,0)\,. \tag{4.123}$$

Therefore $\delta_{\mathbb{C}}(z)$ is a complex delta function, cf. (4.91). Using a linear change of variables one readily verifies

$$\begin{aligned} \delta^K(Ax) &= \frac{\delta^K(x)}{|\det A|} \qquad \text{for} \quad x\in\mathbb{R}^K\,, \quad A\in GL(\mathbb{R}^K)\,,\\ \delta^K_{\mathbb{C}}(Az) &= \frac{\delta^K_{\mathbb{C}}(z)}{\det AA^\dagger} \qquad \text{for} \quad z\in\mathbb{C}^K\,, \quad A\in GL(\mathbb{C}^K)\,. \end{aligned} \tag{4.124}$$

Especially the second line will be used frequently. At times we will also need non-linear coordinate transformations. If we change variables from $x\in\mathbb{R}^K$ to $y(x)\in\mathbb{R}^K$, the measure transforms as $\mathrm{d}^Kx = \mathrm{d}^Ky|\det\frac{\partial x}{\partial y}|$ and the real delta function in the new variables becomes

$$\delta^K(y-y_0) = \left|\det\frac{\partial x}{\partial y}\right|\,\delta^K(x-x_0)\,. \tag{4.125}$$

There is another transformation that we will apply regularly to simplify the argument of the bosonic delta function in the Graßmannian integral (4.111). We want to express a unit vector $\frac{\mathtt{v}}{\|\mathtt{v}\|} \in \mathbb{C}^K$ with $\|\mathtt{v}\| = \sqrt{\mathtt{v}^\dagger \mathtt{v}}$ as a matrix $\mathtt{L} \in U(K)$ acting on a reference vector,

$$\frac{\mathtt{v}}{\|\mathtt{v}\|} = \mathtt{L} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{4.126}$$

A solution to this equation is

$$\mathtt{L} = \frac{1}{\|\mathtt{v}\|} \left(\begin{array}{c:c} \mathtt{v}^1 & -\tilde{\mathtt{v}}^\dagger \\ \hdashline \tilde{\mathtt{v}} & \frac{\overline{\mathtt{v}}^1}{|\mathtt{v}^1|} \left(\|\mathtt{v}\| 1_{n-1} - \frac{\tilde{\mathtt{v}}\tilde{\mathtt{v}}^\dagger}{\|\mathtt{v}\| + |\mathtt{v}^1|} \right) \end{array} \right) , \quad \text{where} \quad \mathtt{v} = \left(\begin{array}{c} \mathtt{v}^1 \\ \hdashline \tilde{\mathtt{v}} \end{array} \right) \tag{4.127}$$

with $\tilde{\mathtt{v}} \in \mathbb{C}^{K-1}$. We note that $\det \mathtt{L} = \left(\frac{\overline{\mathtt{v}}^1}{|\mathtt{v}^1|} \right)^{K-2}$. A different solution of (4.126) is provided by replacing $\mathtt{L} \mapsto \mathtt{L}\,\mathrm{diag}(1, \mathtt{W})$ with any $U(K-1)$ matrix $\mathtt{W}$. See e.g. the discussion of coset spaces of the unitary group in [202]. Notice that for $K = 2$ the matrix in (4.127) becomes

$$\mathtt{L} = \frac{1}{\|\mathtt{v}\|} \begin{pmatrix} \mathtt{v}^1 & -\overline{\mathtt{v}}^2 \\ \mathtt{v}^2 & \overline{\mathtt{v}}^1 \end{pmatrix} . \tag{4.128}$$

4.3.3.2 Two-Site Invariant for $\mathfrak{u}(1,1)$

To begin with, we evaluate the unitary Graßmannian integral (4.111) in the simplest case possible. That is for $(N, K) = (2, 1)$ and the bosonic algebra $\mathfrak{u}(1,1)$. In this case (4.111) becomes a $U(1)$ integral. We employ the parameterization (4.37) of $U(1)$, the Haar measure (4.38) and the integrand $\mathscr{F}(\mathcal{C})$ given in (4.31). Then the integral (4.111) evaluates to

$$\Psi_{2,1}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}) = 2i\, \delta^1(P) \left(-\frac{\lambda_1^1}{\lambda_1^2} \right)^{c_1 - 1} \tag{4.129}$$

with the momentum conserving delta function given in (4.116). To obtain this result we trade the one complex delta function in (4.111) for two real ones. Using (4.125) we change variables such that one real delta function constrains the phase of the original complex argument and the other one the square of its absolute value. The delta function constraining the phase disappears because of the $U(1)$ integral. The other one remains in (4.129) implementing momentum conservation.

4.3.3.3 Four-Site Invariant for $\mathfrak{u}(1,1)$

The $\mathfrak{u}(1,1)$ Yangian invariant with $(N, K) = (4, 2)$ is of importance primarily for two reasons. First, we will learn how to utilize the tool introduced at the end of section 4.3.3.1 for the evaluation of the Graßmannian integral (4.111). Second, the resulting invariant shares a characteristic feature with the six-site invariant of $\mathfrak{u}(2,2)$, which is related to the simplest tree-level NMHV gluon amplitude.

For the case under consideration the Graßmannian integral (4.111) is equipped with the $U(2)$ contour given in (4.39), the corresponding Haar measure (4.41) and the function $\mathscr{F}(\mathcal{C})$ from (4.32). Let us denote the first columns of the matrices $\boldsymbol{\lambda}^{\mathrm{d}}$ and $\boldsymbol{\lambda}^{\mathrm{o}}$ by $\boldsymbol{\lambda}_1^{\mathrm{d}}$ and $\boldsymbol{\lambda}_1^{\mathrm{o}}$, respectively. This notation is superfluous at this point because for $\mathfrak{u}(1,1)$ these matrices consist only of one column. However, it will become necessary in subsequent examples.

For the evaluation of (4.111) we express the vectors $\boldsymbol{\lambda}_1^{\mathrm{d}}$ and $\boldsymbol{\lambda}_1^{\mathrm{o}}$ in terms of $U(2)$ matrices $\mathtt{L}_1^{\mathrm{d}}$ and $\mathtt{L}_1^{\mathrm{o}}$ that obey

$$\mathtt{L}_1^{\mathrm{d}}\begin{pmatrix}1\\0\end{pmatrix}=\frac{\boldsymbol{\lambda}_1^{\mathrm{d}}}{\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|}\,,\qquad \mathtt{L}_1^{\mathrm{o}}\begin{pmatrix}1\\0\end{pmatrix}=\frac{\boldsymbol{\lambda}_1^{\mathrm{o}}}{\|\boldsymbol{\lambda}_1^{\mathrm{o}}\|}\,. \tag{4.130}$$

These matrices are parameterized in terms of the vectors as defined in (4.127) or more explicitly in (4.128). The $U(2)$ integral (4.111) is conveniently addressed after introducing the new integration variable $(\mathtt{L}_1^{\mathrm{d}})^\dagger\mathcal{C}\mathtt{L}_1^{\mathrm{o}}$ and using the left and right invariance of the Haar measure. Employing (4.124), the arguments of the delta functions then become independent of $\boldsymbol{\lambda}^{\mathrm{d}}$ and $\boldsymbol{\lambda}^{\mathrm{o}}$. After parameterizing the new $U(2)$ integration variable as in (4.39), we obtain

$$\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})=4i\,\frac{\delta^1(P)}{\lambda_1^1\overline{\lambda}_1^1+\lambda_1^2\overline{\lambda}_1^2}\,\mathcal{I}(v_1-v_2,c_1,c_2)\,. \tag{4.131}$$

Here we assumed $\lambda_1^1\overline{\lambda}_1^1+\lambda_1^2\overline{\lambda}_1^2\neq 0$ and the momentum conserving delta function is defined in (4.116). The remaining integral is

$$\mathcal{I}(v_1-v_2,c_1,c_2)=\int_0^{2\pi}\mathrm{d}\gamma\,\mathscr{F}(\mathcal{C}(\gamma))\,, \tag{4.132}$$

where the function $\mathscr{F}$ depends on the parameters v_1-v_2,c_1,c_2 and

$$\begin{aligned}\mathcal{C}(\gamma)&=\mathtt{L}_1^{\mathrm{d}}\,\mathrm{diag}(-1,-e^{i\gamma})(\mathtt{L}_1^{\mathrm{o}})^\dagger\\&=\frac{1}{\lambda_1^1\overline{\lambda}_1^1+\lambda_1^2\overline{\lambda}_1^2}\begin{pmatrix}-\lambda_1^1\overline{\lambda}_1^3-\lambda_1^4\overline{\lambda}_1^2e^{i\gamma} & -\lambda_1^1\overline{\lambda}_1^4+\lambda_1^3\overline{\lambda}_1^2e^{i\gamma}\\ -\lambda_1^2\overline{\lambda}_1^3+\lambda_1^4\overline{\lambda}_1^1e^{i\gamma} & -\lambda_1^2\overline{\lambda}_1^4-\lambda_1^3\overline{\lambda}_1^1e^{i\gamma}\end{pmatrix}.\end{aligned} \tag{4.133}$$

This is basically the original unitary integration variable of the Graßmannian integral (4.111) after most of its degrees of freedom have been fixed in terms of the external data $\boldsymbol{\lambda}$ by the delta functions in its integrand. Using the form of $\mathscr{F}$ specified in (4.32), the integral (4.132) becomes

$$\mathcal{I}(v_1-v_2,c_1,c_2)=\int_0^{2\pi}\mathrm{d}\gamma\frac{1}{(e^{i\gamma})^{1-c_2}|\mathtt{A}-e^{i\gamma}\mathtt{B}|^{2(1+v_1-v_2)}(\mathtt{A}-e^{i\gamma}\mathtt{B})^{c_2-c_1}} \tag{4.134}$$

with

$$\mathtt{A}=\frac{\lambda_1^1\overline{\lambda}_1^3}{\lambda_1^1\overline{\lambda}_1^1+\lambda_1^2\overline{\lambda}_1^2}\,,\qquad \mathtt{B}=\frac{-\lambda_1^4\overline{\lambda}_1^2}{\lambda_1^1\overline{\lambda}_1^1+\lambda_1^2\overline{\lambda}_1^2}\,. \tag{4.135}$$

It remains to evaluate the one-dimensional integral (4.134). Focusing on equal representation labels $c_1=c_2$, we rewrite this integral as

$$\mathcal{I}(z,c_1,c_1)=\left||\mathtt{A}|^2-|\mathtt{B}|^2\right|^{-1-z}\int_0^{2\pi}\mathrm{d}\gamma\left(w+\sqrt{w^2-1}\cos(\gamma+\arg(-\overline{\mathtt{A}}\mathtt{B}))\right)^{-1-z}e^{i\gamma(c_1-1)} \tag{4.136}$$

with the variable $w=(|\mathtt{A}|^2+|\mathtt{B}|^2)(||\mathtt{A}|^2-|\mathtt{B}|^2|)^{-1}$ in the range $[1,\infty)$. This formula is independent of the branch of the arg function. We disregard the case $|\mathtt{A}|=|\mathtt{B}|$ from now on. To identify the integral (4.136) with a known function, let us recall some results about the Legendre function $P_\nu^\mu(u)$, where we follow the conventions of [203].[3] It has the integral representation

$$P_\nu^m(u) = \frac{\Gamma(\nu+m+1)}{2\pi\,\Gamma(\nu+1)} \int_{-\pi}^{\pi} \mathrm{d}t\,\big(u+\sqrt{u^2-1}\cos(t)\big)^\nu e^{imt} \tag{4.137}$$

for a non-negative integer m, $\nu \in \mathbb{C}$ and $\operatorname{Re} u > 0$. Furthermore, it obeys

$$P_\nu^m(u) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} P_\nu^{-m}(u)\,, \quad P_\nu^\mu(u) = P_{-\nu-1}^\mu(u) \tag{4.138}$$

with $\mu \in \mathbb{C}$. This function can also be expressed in terms of a hypergeometric function,

$$P_\nu^\mu(u) = 2^{-\nu}(u+1)^{\frac{\mu}{2}+\nu}(u-1)^{-\frac{\mu}{2}} \frac{{}_2F_1(-\nu,-\nu-\mu;1-\mu;\frac{u-1}{u+1})}{\Gamma(1-\mu)} \tag{4.139}$$

for $|u-1| < |u+1|$. Using (4.137) and (4.138), we identify (4.136) with

$$\mathcal{I}(z,c_1,c_1) = \left(\frac{-\bar{\mathtt{A}}\mathtt{B}}{|\mathtt{A}||\mathtt{B}|}\right)^{1-c_1} \frac{2\pi}{\big||\mathtt{A}|^2-|\mathtt{B}|^2\big|^{1+z}} \frac{\Gamma(-z)}{\Gamma(-z+c_1-1)} P_z^{-1+c_1}\left(\frac{|\mathtt{A}|^2+|\mathtt{B}|^2}{\big||\mathtt{A}|^2-|\mathtt{B}|^2\big|}\right). \tag{4.140}$$

This expression inserted into (4.131) is our final form of the four-site Yangian invariant $\Psi_{4,2}(\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}})$ for the algebra $\mathfrak{u}(1,1)$.

The expression (4.140) contains the complex deformation parameter z. In the undeformed limit $z \to 0$ it simplifies using (4.139) together with ${}_2F_1(0,\cdot;\cdot;\cdot) = 1$,

$$\mathcal{I}(0,c_1,c_1) = \left(\frac{\bar{\mathtt{A}}\mathtt{B}}{|\mathtt{A}||\mathtt{B}|}\right)^{1-c_1} \frac{2\pi}{\big||\mathtt{A}|^2-|\mathtt{B}|^2\big|} \left(\frac{|\mathtt{A}|^2+|\mathtt{B}|^2-\big||\mathtt{A}|^2-|\mathtt{B}|^2\big|}{|\mathtt{A}|^2+|\mathtt{B}|^2+\big||\mathtt{A}|^2-|\mathtt{B}|^2\big|}\right)^{\frac{|1-c_1|}{2}}. \tag{4.141}$$

Note that in case of $1-c_1 < 0$, the fraction of gamma functions in (4.140) diverges for $z \to 0$ whereas the Legendre function vanishes. Thus in this case we have to apply the first identity in (4.138) before taking $z \to 0$. Lastly, we can write (4.141) more explicitly as

$$\mathcal{I}(0,c_1,c_1) = \frac{2\pi}{\big||\mathtt{A}|^2-|\mathtt{B}|^2\big|} \begin{cases} \left(\frac{\mathtt{B}}{\mathtt{A}}\right)^{1-c_1} & \text{for} \quad \begin{matrix}|\mathtt{A}|>|\mathtt{B}|,\\ c_1\le 1\end{matrix} \quad \text{or} \quad \begin{matrix}|\mathtt{A}|<|\mathtt{B}|,\\ c_1\ge 1\,,\end{matrix} \\[2ex] \left(\frac{\bar{\mathtt{A}}}{\bar{\mathtt{B}}}\right)^{1-c_1} & \text{for} \quad \begin{matrix}|\mathtt{A}|<|\mathtt{B}|,\\ c_1\le 1\end{matrix} \quad \text{or} \quad \begin{matrix}|\mathtt{A}|>|\mathtt{B}|,\\ c_1\ge 1\,.\end{matrix} \end{cases} \tag{4.142}$$

Notice that for $c_1 = 1$ the expressions for the different cases coincide. In appendix B.2 we study a discrete parity symmetry of the Graßmannian integral (4.111). In case of the invariant $\Psi_{4,2}(\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}})$ for the algebra $\mathfrak{u}(1,1)$ it exchanges the two regions $|\mathtt{A}| > |\mathtt{B}|$ and $|\mathtt{B}| > |\mathtt{A}|$ in (4.142).

We may obtain the result (4.142) for the undeformed case alternatively by rewriting (4.134) as a contour integral in the variable $u = e^{i\gamma}$,

$$\mathcal{I}(0,c_1,c_1) = \frac{i}{\bar{\mathtt{A}}\mathtt{B}} \oint \mathrm{d}u \frac{1}{u^{1-c_1}\left(u-\frac{\mathtt{A}}{\mathtt{B}}\right)\left(u-\frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}\right)}\,, \tag{4.143}$$

[3] These conventions do not comply with the implementation of those functions in the computer algebra program Mathematica [204]. Mathematica can be used after expressing the Legendre functions in terms of hypergeometric ones with (4.139) below.

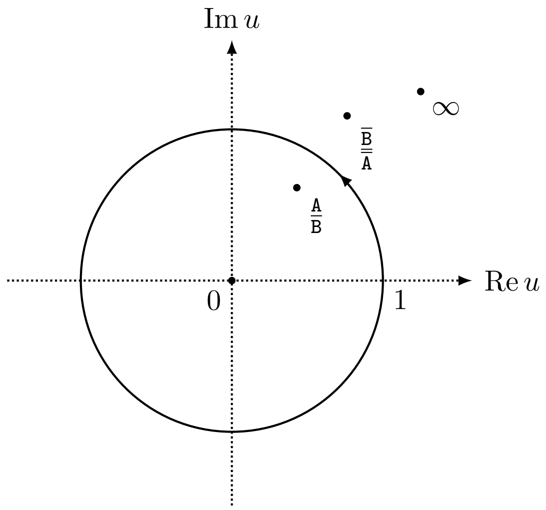


Figure 4.1: The computation of the undeformed Yangian invariant $\Psi_{4,2}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ with $v_1 = v_2$ and $c_1 = c_2$ for $\mathfrak{u}(1,1)$ can be reduced to the one-dimensional complex contour integral (4.143) in the variable u. Its contour is depicted by a solid circle. The integrand has poles whose positions we indicate by a dot. Here we display these for sample external data $\boldsymbol{\lambda}$ satisfying $|\mathtt{A}| < |\mathtt{B}|$. The relation between $\boldsymbol{\lambda}$ and $\mathtt{A}, \mathtt{B}$ is given in (4.135). In case of $|\mathtt{A}| > |\mathtt{B}|$, the pole at $\frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}$ moves inside of the contour and that at $\frac{\mathtt{A}}{\mathtt{B}}$ is outside.

where the contour of integration is the counterclockwise unit circle, see figure 4.1. The integrand can have poles at four points. There can be a pole at $u = 0$, which is inside of the contour, and one at $u = \infty$, which is outside. Furthermore, there is a pair of poles at $u = \frac{\mathtt{A}}{\mathtt{B}}, \frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}$. For all values of $\mathtt{A}$ and $\mathtt{B}$, one of these poles is inside the contour, whereas the other one is outside. Which pole is inside depends on whether $|\mathtt{A}|$ or $|\mathtt{B}|$ is larger, and therefore on the external data $\boldsymbol{\lambda}$, cf. (4.135). In combination, there are always two poles inside the unit circle. The integral (4.143) is then computed easily employing the residue theorem. To obtain (4.142) in this way, we observe that the residues at $u = 0$ and $u = \infty$ vanish for certain values of c_1 and we use that the sum of all residues is equal to zero. We want to emphasize that the varying position of the pair of poles at $u = \frac{\mathtt{A}}{\mathtt{B}}, \frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}$ is the reason for the case distinction in (4.142) for fixed c_1.

Let us comment on the relevance of this calculation for the six-site invariant of $\mathfrak{u}(2,2)$, and thus for the six-particle tree-level NMHV gluon amplitude. In the discussion of that invariant we will encounter an integral which is very similar to (4.143). The main difference being that there will be two pairs of poles instead of only one here.

4.3.3.4 Four-Site Invariant for $\mathfrak{u}(2,2)$ and MHV Gluon Amplitude

Finally, we are ready compute the first amplitude by means of the unitary Graßmannian integral (4.111). We will identify this integral for four sites, the algebra $\mathfrak{u}(2,2)$ and a certain choice of the deformation parameters v_i and the representation labels c_i with the four-particle MHV gluon amplitude $A_{4,2}^{(\text{tree})}$ from (1.15). For general values of v_i and c_i it will be matched with a gluonic component of the deformed superamplitude $\mathcal{A}_{4,2}^{(\text{def.})}$ in (1.48).

For this purpose, we evaluate the integral (4.111) with the $U(2)$ parameterization

(4.39), the Haar measure (4.41) and the function $\mathscr{F}(\mathcal{C})$ given in (4.32). We express the first column $\boldsymbol{\lambda}_1^{\mathrm{d}}$ of the matrix $\boldsymbol{\lambda}^{\mathrm{d}}$ and $\boldsymbol{\lambda}_1^{\mathrm{o}}$ of $\boldsymbol{\lambda}^{\mathrm{o}}$ in terms of the $U(2)$ matrices defined already in (4.130). The delta function in (4.111) that involves $\boldsymbol{\lambda}_1^{\mathrm{d}}$ and $\boldsymbol{\lambda}_1^{\mathrm{o}}$ is then addressed analogously to the discussion of the $\mathfrak{u}(1,1)$ invariant in the previous section 4.3.3.3. This leads to

$$\begin{aligned}\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}}) = 4i\,\frac{\delta(\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|^2-\|\boldsymbol{\lambda}_1^{\mathrm{o}}\|^2)}{\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|^2}\int_0^{2\pi}&\mathrm{d}\gamma\,\mathscr{F}\left(\mathtt{L}_1^{\mathrm{d}}\operatorname{diag}(-1,-e^{i\gamma})(\mathtt{L}_1^{\mathrm{o}})^\dagger\right)\\ &\cdot\delta_{\mathbb{C}}^2\left((\mathtt{L}_1^{\mathrm{d}})^\dagger\boldsymbol{\lambda}_2^{\mathrm{d}}+\operatorname{diag}(-1,-e^{i\gamma})(\mathtt{L}_1^{\mathrm{o}})^\dagger\boldsymbol{\lambda}_2^{\mathrm{o}}\right)\,,\end{aligned}\tag{4.144}$$

where we assumed $\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|\neq 0$. The evaluation of the remaining integral yields

$$\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}}) = 8i\,\delta^4(P)\mathscr{F}(\mathcal{C})\tag{4.145}$$

with

$$\mathcal{C}=\mathtt{L}_1^{\mathrm{d}}\operatorname{diag}\left(-1,\frac{\langle 21\rangle}{\langle 34\rangle}\right)(\mathtt{L}_1^{\mathrm{o}})^\dagger\tag{4.146}$$

and the momentum conserving delta function (4.116) in the form

$$\delta^4(P)=\delta(\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|^2-\|\boldsymbol{\lambda}_1^{\mathrm{o}}\|^2)\delta(\|\boldsymbol{\lambda}_2^{\mathrm{d}}\|^2-\|\boldsymbol{\lambda}_2^{\mathrm{o}}\|^2)\delta_{\mathbb{C}}((\boldsymbol{\lambda}_1^{\mathrm{d}})^\dagger\boldsymbol{\lambda}_2^{\mathrm{d}}-(\boldsymbol{\lambda}_1^{\mathrm{o}})^\dagger\boldsymbol{\lambda}_2^{\mathrm{o}})\,.\tag{4.147}$$

Here we used (4.125) to manipulate the delta function in (4.144). In particular, the absolute value and the first component of the $\mathbb{C}^2$-vector in its argument directly lead to the delta functions in (4.147). Moreover, the phase $e^{i\gamma}$ is fixed by the remaining integral in (4.144) and expressed in terms of the angle bracket defined in (1.8).

As a brief interlude, we recall for convenience the definition of said bracket from (1.8),

$$\langle ij\rangle=\lambda_1^i\lambda_2^j-\lambda_2^i\lambda_1^j\,.\tag{4.148}$$

Let us also recapitulate the definition of the square bracket from that equation,

$$[ij]=-\tilde\lambda_1^i\tilde\lambda_2^j+\tilde\lambda_2^i\tilde\lambda_1^j\,.\tag{4.149}$$

Using the reality conditions for the spinor helicity variables in (4.113) we obtain, cf. (1.10),

$$[kk']=-\overline{\langle kk'\rangle}\,,\quad [ll']=-\overline{\langle ll'\rangle}\,,\quad [kl]=\overline{\langle kl\rangle}\tag{4.150}$$

for dual sites $k,k'=1,\dots,K$ and ordinary sites $l,l'=K+1,\dots,N$. The brackets obey the Schouten identity, cf. (1.12),

$$\langle ij\rangle\langle kl\rangle-\langle ik\rangle\langle jl\rangle=\langle il\rangle\langle kj\rangle\,.\tag{4.151}$$

Furthermore, due to momentum conservation (4.147) we have

$$\sum_{k=1}^{K}\langle ik\rangle\overline{\langle kj\rangle}=\sum_{l=K+1}^{N}\langle il\rangle\overline{\langle lj\rangle}\quad\text{or equivalently}\quad\sum_{h=1}^{N}\langle ih\rangle[hj]=0\,,\tag{4.152}$$

where i and j are can denote any site. The latter from of this condition was presented already in (1.13). As indicated by the notation, the formulas in this paragraph are valid for general N and K. We will utilize them also for further sample Yangian invariants below.

Let us return to the evaluation of $\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$. Note that using (4.152) the combination of brackets in (4.146), which originated from $e^{i\gamma}$, is indeed a phase. Moreover, with (4.151) and (4.152) we obtain for the entire matrix in (4.146)

$$\mathcal{C}=\frac{1}{\langle 34\rangle}\begin{pmatrix}\langle 41\rangle & \langle 13\rangle\\ \langle 42\rangle & \langle 23\rangle\end{pmatrix}=\frac{1}{\overline{\langle 12\rangle}}\begin{pmatrix}\overline{\langle 23\rangle} & \overline{\langle 24\rangle}\\ \overline{\langle 31\rangle} & \overline{\langle 41\rangle}\end{pmatrix} \tag{4.153}$$

and verify the unitarity of this matrix. Finally, we insert the function $\mathscr{F}$ from (4.32) into (4.145) to obtain the Yangian invariant with $(N,K)=(4,2)$ for the algebra $\mathfrak{u}(2,2)$,

$$\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})=8i\,\delta^4(P)\frac{\langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(\frac{\langle 14\rangle}{\langle 34\rangle}\right)^{c_1}\left(\frac{\langle 12\rangle}{\langle 14\rangle}\right)^{c_2}\left(\frac{\langle 34\rangle\overline{\langle 34\rangle}}{\langle 14\rangle\overline{\langle 14\rangle}}\right)^{v_1-v_2}. \tag{4.154}$$

We conclude by identifying this invariant with known results. We start by focusing on the representation labels $c_1=c_2=0$. According to (4.75) the four sites then carry the helicities $(+1,+1,-1,-1)$. If in addition $v_1=v_2$, the invariant reduces, up to a prefactor, to the well-known Parke-Taylor formula (1.15) for the tree-level MHV gluon amplitude $A_{4,2}^{(\text{tree})}$. We move on the discuss the case of general deformation parameters $c_1,c_2\in\mathbb{Z}$ and $v_1,v_2\in\mathbb{C}$. Observe that the invariant (4.154) is a single-valued function in λ_α^i because the complex deformation parameters appear only as the exponent of a non-negative real number. This property can be traced back to the single-valuedness of the integrand of the unitary Graßmannian formula discussed in section 4.1.4.1. To identify (4.154) with a deformed amplitude from the introductory section 1.3.6, we have to give up the manifest single-valuedness. Employing the momentum conservation in (4.152), we write

$$\frac{\langle 34\rangle\overline{\langle 34\rangle}}{\langle 14\rangle\overline{\langle 14\rangle}}=\frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 41\rangle}. \tag{4.155}$$

This equality shows that (4.154) is, up to a numerical constant, a gluonic component $A_{4,2}^{(\text{def.})}$ of the deformed superamplitude $\mathcal{A}_{4,2}^{(\text{def.})}$ given in (1.48). Merely the deformation parameters v_1,v_2 are parameterized in a slightly different way in that formula.

4.3.3.5 Six-Site Invariant for $\mathfrak{u}(2,2)$ and NMHV Gluon Amplitude

After the successful identification of the first amplitude, we move on to the evaluation of the unitary Graßmannian integral (4.111) for six sites and the algebra $\mathfrak{u}(2,2)$. From the usual Graßmannian integral approach to amplitudes reviewed in section 1.3.5 we expect our invariant to be related to the six-particle NMHV gluon amplitude $A_{6,3}^{(\text{tree})}$. This is a crucial test of our method because it is the first instance where one has to fix a integration contour “by hand” in the usual approach. We do not have this freedom with our unitary contour. Therefore it has to produce the correct result automatically. Furthermore, the unitary Graßmannian integral (4.111) naturally includes deformation parameters. Thus we may also hope to relate it to the sought-after deformed gluon amplitude $A_{6,3}^{(\text{def.})}$. Recall from section 1.3.6 that the construction of the deformed superamplitude $\mathcal{A}_{6,3}^{(\text{def.})}$, and thus also of $A_{6,3}^{(\text{def.})}$, is still an open problem.

We want to compute the Graßmannian integral (4.111) for $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ with the $U(3)$ contour (4.42), the associated Haar measure (4.46) and the function $\mathscr{F}(\mathcal{C})$ in (4.33). To address those delta functions in the Graßmannian integral involving the first column $\boldsymbol{\lambda}_1^{\text{d}}$ of the matrix $\boldsymbol{\lambda}^{\mathrm{d}}$ and $\boldsymbol{\lambda}_1^{\mathrm{o}}$ of $\boldsymbol{\lambda}^{\mathrm{o}}$, we introduce $U(3)$ matrices $\mathtt{L}_1^{\mathrm{d}}$ and $\mathtt{L}_1^{\mathrm{o}}$ that satisfy

$$\mathtt{L}_1^{\mathrm{d}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\boldsymbol{\lambda}_1^{\mathrm{d}}}{\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|}, \quad \mathtt{L}_1^{\mathrm{o}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\boldsymbol{\lambda}_1^{\mathrm{o}}}{\|\boldsymbol{\lambda}_1^{\mathrm{o}}\|}. \tag{4.156}$$

These matrices are parameterized in terms of the column vectors as defined in (4.127). Then we change the integration variable of the $U(3)$ Graßmannian integral from $\mathcal{C}$ to $(\mathtt{L}_1^{\mathrm{d}})^\dagger \mathcal{C}\, \mathtt{L}_1^{\mathrm{o}}$ and parameterize this new variable as in (4.42). The delta functions containing $\boldsymbol{\lambda}_1^{\mathrm{d}}$ and $\boldsymbol{\lambda}_1^{\mathrm{o}}$ then fix parts of the integration variable such that we are left with the $U(2)$ integral

$$\begin{aligned} \Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}) = 8 &\frac{\delta(\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|^2 - \|\boldsymbol{\lambda}_1^{\mathrm{o}}\|^2)}{\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|^4} \delta_{\mathbb{C}}([(\mathtt{L}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}]^1 - [(\mathtt{L}_1^{\mathrm{o}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{o}}]^1)(\chi_2 i)^{-1} \\ &\cdot \int_{U(2)} [\mathrm{d}\mathcal{D}]\, \mathscr{F}\left(\mathtt{L}_1^{\mathrm{d}} \operatorname{diag}(-1, \mathcal{D})(\mathtt{L}_1^{\mathrm{o}})^\dagger\right) \delta^2_{\mathbb{C}}([(\mathtt{L}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}]^{\tilde{}} + \mathcal{D}[(\mathtt{L}_1^{\mathrm{o}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{o}}]^{\tilde{}}), \end{aligned} \tag{4.157}$$

where we assumed $\|\boldsymbol{\lambda}_1^{\mathrm{d}}\| \neq 0$. Here, as in (4.127), $[(\mathtt{L}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}]^1$ denotes the first component of the vector $(\mathtt{L}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}$ and $[(\mathtt{L}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}]^{\tilde{}}$ refers to its remaining two components. To proceed with the $U(2)$ integral, we introduce the $U(2)$ matrices $\mathtt{L}_2^{\mathrm{d}}$ and $\mathtt{L}_2^{\mathrm{o}}$ obeying

$$\mathtt{L}_2^{\mathrm{d}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{[(\mathtt{L}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}]^{\tilde{}}}{\|[(\mathtt{L}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}]^{\tilde{}}\|}, \quad \mathtt{L}_2^{\mathrm{o}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{[(\mathtt{L}_1^{\mathrm{o}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{o}}]^{\tilde{}}}{\|[(\mathtt{L}_1^{\mathrm{o}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{o}}]^{\tilde{}}\|}. \tag{4.158}$$

These matrices are parametrized as shown in (4.128). Then the $U(2)$ integral in (4.157) is evaluated after changing variables from $\mathcal{D}$ to $(\mathtt{L}_2^{\mathrm{d}})^\dagger \mathcal{D}\, \mathtt{L}_2^{\mathrm{o}}$. This yields

$$\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}) = 32 \frac{\delta^4(P)}{s_{123}} \mathcal{I}(v_1 - v_2, v_2 - v_3, c_1, c_2, c_3) \tag{4.159}$$

with the momentum conserving delta function defined in (4.116). Furthermore, we introduced

$$\begin{aligned} s_{123} &= \|\boldsymbol{\lambda}_1^{\mathrm{d}}\|^2 \|[(\mathtt{L}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}]^{\tilde{}}\|^2 = \|\boldsymbol{\lambda}_1^{\mathrm{d}}\|^2 \|\boldsymbol{\lambda}_2^{\mathrm{d}}\|^2 - (\boldsymbol{\lambda}_1^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_2^{\mathrm{d}}\, (\boldsymbol{\lambda}_2^{\mathrm{d}})^\dagger \boldsymbol{\lambda}_1^{\mathrm{d}} \\ &= \langle 12 \rangle \overline{\langle 12 \rangle} + \langle 13 \rangle \overline{\langle 13 \rangle} + \langle 23 \rangle \overline{\langle 23 \rangle}. \end{aligned} \tag{4.160}$$

Analogously one defines s_{456} in terms of $\boldsymbol{\lambda}^{\mathrm{o}}$. Due to momentum conservation (4.152) we have $s_{123} = s_{456}$. In (4.159) there remains the $U(1)$ integral

$$\mathcal{I}(v_1 - v_2, v_2 - v_3, c_1, c_2, c_3) = \int_0^{2\pi} \mathrm{d}\gamma\, \mathscr{F}\left(\mathcal{C}(\gamma)\right), \tag{4.161}$$

where the integral depends on the parameters v_i and c_i through the function $\mathscr{F}$ and we introduced the unitary matrix

$$\mathcal{C}(\gamma) = \mathtt{L}_1^{\mathrm{d}} \operatorname{diag}(1, \mathtt{L}_2^{\mathrm{d}}) \operatorname{diag}(-1_2, -e^{i\gamma}) \big(\mathtt{L}_1^{\mathrm{o}} \operatorname{diag}(1, \mathtt{L}_2^{\mathrm{o}})\big)^\dagger. \tag{4.162}$$

In essence, this is the integration variable of the original Graßmannian integral (4.111) after the $U(3)$ degrees of freedom, except for one $U(1)$ phase, have been fixed by the delta functions in the integrand.

In order to obtain an explicit representation of the integral (4.161) we compute

$$\mathtt{L}_1^{\mathrm{d}}\,\mathrm{diag}(1,\mathtt{L}_2^{\mathrm{d}}) = \begin{pmatrix} \frac{\lambda_1^1}{\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|} & \frac{-\bar{\lambda}_1^2\langle 12\rangle-\bar{\lambda}_1^3\langle 13\rangle}{\sqrt{s_{123}}\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|} & \frac{\bar{\lambda}_1^1\overline{\langle 23\rangle}}{\sqrt{s_{123}}|\lambda_1^1|} \\ \frac{\lambda_1^2}{\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|} & \frac{\bar{\lambda}_1^1\langle 12\rangle-\bar{\lambda}_1^3\langle 23\rangle}{\sqrt{s_{123}}\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|} & \frac{\bar{\lambda}_1^1\overline{\langle 31\rangle}}{\sqrt{s_{123}}|\lambda_1^1|} \\ \frac{\lambda_1^3}{\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|} & \frac{\bar{\lambda}_1^1\langle 13\rangle+\bar{\lambda}_1^2\langle 23\rangle}{\sqrt{s_{123}}\|\boldsymbol{\lambda}_1^{\mathrm{d}}\|} & \frac{\bar{\lambda}_1^1\overline{\langle 12\rangle}}{\sqrt{s_{123}}|\lambda_1^1|} \end{pmatrix}. \tag{4.163}$$

The matrix $\mathtt{L}_1^{\mathrm{o}}\,\mathrm{diag}(1,\mathtt{L}_2^{\mathrm{o}})$ is of the same form with $\boldsymbol{\lambda}_1^{\mathrm{d}}$ replaced by $\boldsymbol{\lambda}_1^{\mathrm{o}}$ and the site indices $1,2,3$ by $4,5,6$, respectively. Using the Schouten identity (4.151) and momentum conservation (4.152), we obtain

$$\begin{aligned} \mathcal{C}(\gamma) = {} & \frac{1}{s_{123}}\begin{pmatrix} \langle 1|2+3|\overline{4}\rangle & \langle 1|2+3|\overline{5}\rangle & \langle 1|2+3|\overline{6}\rangle \\ \langle 2|1+3|\overline{4}\rangle & \langle 2|1+3|\overline{5}\rangle & \langle 2|1+3|\overline{6}\rangle \\ \langle 3|1+2|\overline{4}\rangle & \langle 3|1+2|\overline{5}\rangle & \langle 3|1+2|\overline{6}\rangle \end{pmatrix} \\ & - \frac{1}{s_{123}} e^{i\gamma} \frac{\bar{\lambda}_1^1}{|\lambda_1^1|}\frac{\lambda_1^4}{|\lambda_1^4|} \begin{pmatrix} \langle 56\rangle\overline{\langle 23\rangle} & \langle 64\rangle\overline{\langle 23\rangle} & \langle 45\rangle\overline{\langle 23\rangle} \\ \langle 56\rangle\overline{\langle 31\rangle} & \langle 64\rangle\overline{\langle 31\rangle} & \langle 45\rangle\overline{\langle 31\rangle} \\ \langle 56\rangle\overline{\langle 12\rangle} & \langle 64\rangle\overline{\langle 12\rangle} & \langle 45\rangle\overline{\langle 12\rangle} \end{pmatrix}, \end{aligned} \tag{4.164}$$

where we employed the shorthand notation $\langle 1|2+3|\overline{4}\rangle = \langle 12\rangle\overline{\langle 24\rangle} + \langle 13\rangle\overline{\langle 34\rangle}$ etc. We work with the complex conjugate $\overline{\langle ij\rangle}$ of the angle brackets instead of the square brackets $[ij]$ to make the analytic structure of the expressions more transparent. One can easily translate between the two with the help of (4.150). According to (4.33) the function $\mathscr{F}$ in (4.161) involves the minors

$$\begin{aligned} \det\mathcal{C}(\gamma) &= -e^{i\gamma}\frac{\bar{\lambda}_1^1}{|\lambda_1^1|}\frac{\lambda_1^4}{|\lambda_1^4|}, \\ [1]_{\mathcal{C}(\gamma)} &= \mathtt{A} - e^{i\gamma}\frac{\bar{\lambda}_1^1}{|\lambda_1^1|}\frac{\lambda_1^4}{|\lambda_1^4|}\mathtt{B} \quad \text{with} \quad \mathtt{A} = \frac{\langle 1|2+3|\overline{4}\rangle}{s_{123}}, \quad \mathtt{B} = \frac{\langle 56\rangle\overline{\langle 23\rangle}}{s_{123}}, \\ [3]_{\mathcal{C}(\gamma)} &= \bar{\mathtt{C}} - e^{i\gamma}\frac{\bar{\lambda}_1^1}{|\lambda_1^1|}\frac{\lambda_1^4}{|\lambda_1^4|}\bar{\mathtt{D}} \quad \text{with} \quad \mathtt{C} = \frac{\overline{\langle 3|1+2|6\rangle}}{s_{123}}, \quad \mathtt{D} = \frac{\overline{\langle 45\rangle}\langle 12\rangle}{s_{123}}. \end{aligned} \tag{4.165}$$

With these we can provide the desired explicit expression for the $U(1)$ integral (4.161),

$$\begin{aligned} \mathcal{I}(z_1,z_2,c_1,c_2,c_3) = \int_0^{2\pi}\mathrm{d}\gamma\, & \frac{1}{(-e^{i\gamma})^{2-c_2}|\mathtt{A}-e^{i\gamma}\mathtt{B}|^{2(1+z_1)}(\mathtt{A}-e^{i\gamma}\mathtt{B})^{c_2-c_1}} \\ & \cdot \frac{1}{|\mathtt{C}-e^{-i\gamma}\mathtt{D}|^{2(1+z_2)}(\mathtt{C}-e^{-i\gamma}\mathtt{D})^{c_3-c_2}}, \end{aligned} \tag{4.166}$$

where we shifted the integration variable by the phase appearing e.g. in the first line of (4.165). Let us comment that we were able to identify a similar integral in (4.134), which occurred for the invariant $\Psi_{4,2}(\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}})$ of $\mathfrak{u}(1,1)$, with a Legendre function in (4.140). It would be desirable to also understand (4.166) in terms of a known special function. Returning to our main discussion, equation (4.159) together with (4.166) is our final form of the invariant $\Psi_{6,3}(\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}})$ for $\mathfrak{u}(2,2)$. This result immediately raises a pressing question. Is $\Psi_{6,3}(\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}})$ the sought-after deformed gluon amplitude $A_{6,3}^{(\mathrm{def.})}$? It certainly contains the deformation parameters $v_1,v_2,v_3\in\mathbb{C}$ and $c_1,c_2,c_3\in\mathbb{Z}$. What is more, it is Yangian invariant because of the unitary contour in the Graßmannian integral (4.111). From the perspective of the introductory section 1.3.6, the combination of these two properties is already highly non-trivial. Recall that the integrand of the deformed Graßmannian formula

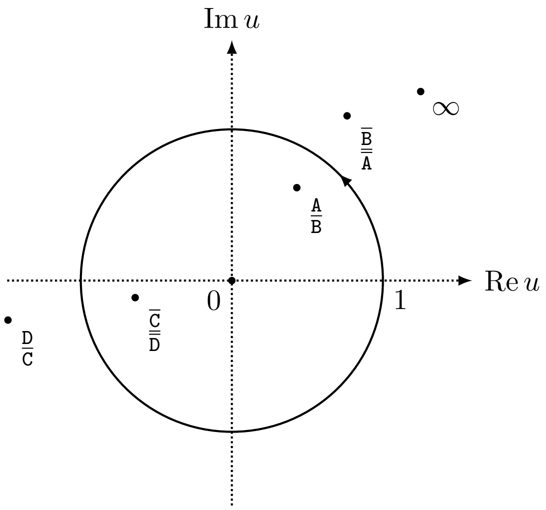


Figure 4.2: The evaluation of the invariant $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ for $\mathfrak{u}(2,2)$ with the parameters $v_1 = v_2 = v_3 \in \mathbb{C}$, $c_1 = c_2 = c_3 \in \mathbb{Z}$ yields the complex contour integral (4.167) in the variable u. The positions of the pairs of poles at $u = \frac{\mathtt{A}}{\mathtt{B}}, \frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}$ and $u = \frac{\mathtt{D}}{\mathtt{C}}, \frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}$ depend on the external data $\boldsymbol{\lambda}$, cf. (4.165). Exactly one pole of each pair is inside the contour. This divides the external data into four regions controlled by $s_{234}, s_{126} \gtrless 0$, see (4.170). The pole configuration for the region with $s_{234}, s_{126} > 0$ is depicted here. For this region $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ matches the amplitude $A_{6,3}^{(\text{tree})}$ as shown in (4.172).

(1.54) has branch cuts, which make the choice of a closed contour guaranteeing Yangian invariance difficult.

To approach the above question we have to investigate the undeformed limit of $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ and relate it to the gluon amplitude $A_{6,3}^{(\text{tree})}$. To this end, we study the integral (4.166) for the special case with $z_1 = z_2 = 0$ and $c_1 = c_2 = c_3$. We rewrite it as the complex contour integral

$$
\begin{aligned}
\mathcal{I}(0,0,c_1,c_1,c_1) &= \frac{i}{\bar{\mathtt{A}}\mathtt{B}\mathtt{C}\bar{\mathtt{D}}} \oint \mathrm{d}u \, \frac{1}{(-u)^{1-c_1}\left(u - \frac{\mathtt{A}}{\mathtt{B}}\right)\left(u - \frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}\right)\left(u - \frac{\mathtt{D}}{\mathtt{C}}\right)\left(u - \frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}\right)} \\
&= \oint \mathrm{d}u \, \mathscr{I}(u) \,,
\end{aligned}
\tag{4.167}
$$

where we integrate counterclockwise along the unit circle, see figure 4.2. For later use, we introduced the symbol $\mathscr{I}(u)$ for the integrand. Notice the striking similarity to the integral (4.143), which appears for the invariant $\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ of $\mathfrak{u}(1,1)$. Consequently these integrals share some key features. The integrand of (4.167) can have a pole at $u = 0$ inside of the contour and at $u = \infty$ outside of it. In addition, there are two pairs of poles at $u = \frac{\mathtt{A}}{\mathtt{B}}, \frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}$ and at $u = \frac{\mathtt{D}}{\mathtt{C}}, \frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}$. The positions of these poles depend on the external data $\boldsymbol{\lambda}$, cf. (4.165). Interestingly, for generic external data with $|\mathtt{A}| \neq |\mathtt{B}|$ and $|\mathtt{C}| \neq |\mathtt{D}|$, one pole of each pair is always inside of the contour and the other one is outside. Hence the contour in (4.167) encircles three poles. We want to evaluate this integral using the residue theorem.

The residues are conveniently expressed in terms of

$$
\begin{aligned}
|\mathtt{B}|^2 - |\mathtt{A}|^2 &= \frac{s_{234}}{s_{123}}\,, \qquad & |\mathtt{D}|^2 - |\mathtt{C}|^2 &= \frac{s_{126}}{s_{123}}\,,\\
\mathtt{A}\bar{\mathtt{D}} - \mathtt{B}\bar{\mathtt{C}} &= \frac{\langle 5|1-6\overline{|2\rangle}}{s_{123}}\,, \qquad & \mathtt{BD} - \mathtt{AC} &= \frac{\langle 16\rangle\overline{\langle 34\rangle}}{s_{123}}\,,
\end{aligned}
\tag{4.168}
$$

where $s_{234} = \langle 23\rangle\overline{\langle 23\rangle} - \langle 24\rangle\overline{\langle 24\rangle} - \langle 34\rangle\overline{\langle 34\rangle} = \langle 23\rangle[32] + \langle 24\rangle[42] + \langle 34\rangle[43]$ and s_{126} is defined analogously with $2,3,4$ in s_{234} replaced by $1,2,6$, respectively. We used the Schouten identity (4.151) and momentum conservation (4.152) to derive (4.168) from the expressions for $\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{D}$ in (4.165). The residues of the integrand in (4.167) then read

$$
\begin{aligned}
\mathrm{res}_0 \mathscr{I}(u) &= \begin{cases} 0 & \text{for} \quad c_1 \geq 1\,,\\ -i\dfrac{s_{123}^4}{\langle 1|2+3\overline{|4\rangle}\,\overline{\langle 56\rangle}\langle 23\rangle\langle 3|1+2\overline{|6\rangle}\,\overline{\langle 45\rangle}\langle 12\rangle} & \text{for} \quad c_1 = 0\,,\end{cases}\\
\mathrm{res}_\infty \mathscr{I}(u) &= 0 \quad \text{for} \quad c_1 \leq 3\,,\\
\mathrm{res}_{\frac{\mathtt{A}}{\mathtt{B}}} \mathscr{I}(u) &= -i\frac{s_{123}}{\langle 5|1-6\overline{|2\rangle}}\frac{\langle 1|2+3\overline{|4\rangle}\langle 56\rangle\overline{\langle 23\rangle}}{s_{234}\langle 16\rangle\overline{\langle 34\rangle}}\left(-\frac{\langle 1|2+3\overline{|4\rangle}}{\langle 56\rangle\overline{\langle 23\rangle}}\right)^{c_1-2}\,,\\
\mathrm{res}_{\frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}} \mathscr{I}(u) &= i\frac{s_{123}}{\overline{\langle 5|}1-6|2\rangle}\frac{\overline{\langle 1|}2+3|4\rangle\overline{\langle 56\rangle}\langle 23\rangle}{s_{234}\overline{\langle 16\rangle}\langle 34\rangle}\left(-\frac{\overline{\langle 1|}2+3|4\rangle}{\overline{\langle 56\rangle}\langle 23\rangle}\right)^{2-c_1}\,,\\
\mathrm{res}_{\frac{\mathtt{D}}{\mathtt{C}}} \mathscr{I}(u) &= -i\frac{s_{123}}{\overline{\langle 5|}1-6|2\rangle}\frac{\overline{\langle 3|}1+2|6\rangle\overline{\langle 45\rangle}\langle 12\rangle}{s_{126}\overline{\langle 34\rangle}\langle 16\rangle}\left(-\frac{\overline{\langle 3|}1+2|6\rangle}{\overline{\langle 45\rangle}\langle 12\rangle}\right)^{2-c_1}\,,\\
\mathrm{res}_{\frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}} \mathscr{I}(u) &= i\frac{s_{123}}{\langle 5|1-6\overline{|2\rangle}}\frac{\langle 3|1+2\overline{|6\rangle}\langle 45\rangle\overline{\langle 12\rangle}}{s_{126}\langle 34\rangle\overline{\langle 16\rangle}}\left(-\frac{\langle 3|1+2\overline{|6\rangle}}{\langle 45\rangle\overline{\langle 12\rangle}}\right)^{c_1-2}\,.
\end{aligned}
\tag{4.169}
$$

Employing the residue theorem, the integral (4.167) becomes

$$
\mathcal{I}(0,0,c_1,c_1,c_1) = 2\pi i \begin{cases}
\mathrm{res}_0\mathscr{I}(u) + \mathrm{res}_{\frac{\mathtt{A}}{\mathtt{B}}}\mathscr{I}(u) + \mathrm{res}_{\frac{\mathtt{D}}{\mathtt{C}}}\mathscr{I}(u) & \text{for} \quad \begin{matrix} s_{234} > 0\,,\\ s_{126} < 0\,,\end{matrix}\\
\mathrm{res}_0\mathscr{I}(u) + \mathrm{res}_{\frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}}\mathscr{I}(u) + \mathrm{res}_{\frac{\mathtt{D}}{\mathtt{C}}}\mathscr{I}(u) & \text{for} \quad \begin{matrix} s_{234} < 0\,,\\ s_{126} < 0\,,\end{matrix}\\
\mathrm{res}_0\mathscr{I}(u) + \mathrm{res}_{\frac{\mathtt{A}}{\mathtt{B}}}\mathscr{I}(u) + \mathrm{res}_{\frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}}\mathscr{I}(u) & \text{for} \quad \begin{matrix} s_{234} > 0\,,\\ s_{126} > 0\,,\end{matrix}\\
\mathrm{res}_0\mathscr{I}(u) + \mathrm{res}_{\frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}}\mathscr{I}(u) + \mathrm{res}_{\frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}}\mathscr{I}(u) & \text{for} \quad \begin{matrix} s_{234} < 0\,,\\ s_{126} > 0\,.\end{matrix}
\end{cases}
\tag{4.170}
$$

The four kinematic regions result from the two pairs of poles explained after the contour integral formula (4.167). Which pole of the pair at $u = \frac{\mathtt{A}}{\mathtt{B}}, \frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}$ is inside of the contour is determined by $\frac{|\mathtt{A}|}{|\mathtt{B}|} \lessgtr 1$. Here we used (4.168) to translate this into the condition $s_{234} \gtrless 0$. Notice from (4.160) that $s_{123} > 0$ cannot change the sign because of the reality conditions on the spinor helicity variables. Analogously, for the pair of poles at $u = \frac{\mathtt{D}}{\mathtt{C}}, \frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}$, we write the condition $\frac{|\mathtt{C}|}{|\mathtt{D}|} \gtrless 1$ as $s_{126} \lessgtr 0$. The formula (4.159) combined with (4.170) is the final result for the invariant $\Psi_{6,3}(\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}})$ with trivial complex deformation parameters $v_1 = v_2 = v_3$.

Let us add an aside. As the reader might have already noted, we did not present the residues $\mathrm{res}_0\mathscr{I}(u)$ and $\mathrm{res}_\infty\mathscr{I}(u)$ in (4.169) for the entire range of the representation label $c_1 \in \mathbb{Z}$. However, using that the sum of all residues vanishes, we can still express the result

(4.170) for all $c_1 \in \mathbb{Z}$ in terms of those residue that we computed explicitly. This yields

$$
\begin{aligned}
&\mathcal{I}(0,0,c_1,c_1,c_1) = \\
&\cdot\, 2\pi i \frac{s_{234}}{|s_{234}|}
\begin{cases}
\operatorname{res}_{\frac{\mathtt{A}}{\mathtt{B}}} \mathscr{I}(u) + \operatorname{res}_{\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}} \mathscr{I}(u) & \text{for} \quad \begin{array}{r} s_{234} < 0\,, \\ s_{126} < 0\,, \\ c_1 \leq 3 \end{array} \quad \text{or} \quad \begin{array}{r} s_{234} > 0\,, \\ s_{126} > 0\,, \\ c_1 \geq 1\,, \end{array} \\[2ex]
-\operatorname{res}_{\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}} \mathscr{I}(u) - \operatorname{res}_{\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}} \mathscr{I}(u) & \text{for} \quad \begin{array}{r} s_{234} > 0\,, \\ s_{126} < 0\,, \\ c_1 \leq 3 \end{array} \quad \text{or} \quad \begin{array}{r} s_{234} < 0\,, \\ s_{126} > 0\,, \\ c_1 \geq 1\,, \end{array} \\[2ex]
\operatorname{res}_{\frac{\mathtt{A}}{\mathtt{B}}} \mathscr{I}(u) + \operatorname{res}_{\frac{\mathtt{D}}{\mathtt{C}}} \mathscr{I}(u) & \text{for} \quad \begin{array}{r} s_{234} < 0\,, \\ s_{126} > 0\,, \\ c_1 \leq 3 \end{array} \quad \text{or} \quad \begin{array}{r} s_{234} > 0\,, \\ s_{126} < 0\,, \\ c_1 \geq 1\,, \end{array} \\[2ex]
-\operatorname{res}_{\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}} \mathscr{I}(u) - \operatorname{res}_{\frac{\mathtt{D}}{\mathtt{C}}} \mathscr{I}(u) & \text{for} \quad \begin{array}{r} s_{234} > 0\,, \\ s_{126} > 0\,, \\ c_1 \leq 3 \end{array} \quad \text{or} \quad \begin{array}{r} s_{234} < 0\,, \\ s_{126} < 0\,, \\ c_1 \geq 1\,. \end{array}
\end{cases}
\end{aligned}
\tag{4.171}
$$

We remark that the number of cases reduces for $c_1 = 1, 2, 3$.

Returning to our objective of relating the undeformed invariant $\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ to the gluon amplitude $A_{6,3}^{(\text{tree})}$, we choose the representation labels $c_1 = c_2 = c_3 = 0$ in addition to $v_1 = v_2 = v_3$. The six sites of the invariant then carry the helicities $(+1, +1, +1, -1, -1, -1)$, cf. (4.75). The expressions for the invariant in (4.170) and (4.171) reduce in the kinematic region $s_{234}, s_{126} > 0$ to

$$
\begin{aligned}
\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}) &= 64\pi i \delta^4(P) \frac{\operatorname{res}_0 \mathscr{I}(u) + \operatorname{res}_{\frac{\mathtt{A}}{\mathtt{B}}} \mathscr{I}(u) + \operatorname{res}_{\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}} \mathscr{I}(u)}{s_{123}} \\
&= 64\pi i \delta^4(P) \frac{-\operatorname{res}_{\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}} \mathscr{I}(u) - \operatorname{res}_{\frac{\mathtt{D}}{\mathtt{C}}} \mathscr{I}(u)}{s_{123}} \\
&= 64\pi \delta^4(P) \frac{1}{\langle 5|1-6|2\rangle} \left(\frac{\langle 6|1+2\overline{|3\rangle}^3}{\langle 61\rangle\langle 12\rangle\overline{\langle 34\rangle}\,\overline{\langle 45\rangle} s_{126}} - \frac{\langle 4|5+6\overline{|1\rangle}^3}{\langle 23\rangle\langle 34\rangle\overline{\langle 16\rangle}\,\overline{\langle 65\rangle} s_{156}} \right) \\
&= 64\pi \delta^4(P) \frac{1}{[5|1+6|2\rangle} \left(\frac{\langle 6|1+2|3]^3}{\langle 61\rangle\langle 12\rangle[34][45] s_{126}} + \frac{\langle 4|5+6|1]^3}{\langle 23\rangle\langle 34\rangle[16][65] s_{156}} \right).
\end{aligned}
\tag{4.172}
$$

Here $s_{156} = -\langle 15\rangle\overline{\langle 15\rangle} - \langle 16\rangle\overline{\langle 16\rangle} + \langle 56\rangle\overline{\langle 56\rangle} = \langle 15\rangle[51] + \langle 16\rangle[61] + \langle 56\rangle[65] = s_{234}$. This formula is proportional to the six-particle NMHV gluon amplitude $A_{6,3}^{(\text{tree})}$ from (1.17). Hence in this one kinematic region the $U(3)$ contour we started out with in the Graßmannian integral (4.111) automatically selects the desired three residues out of six. Curiously, our formula seems to differ from $A_{6,3}^{(\text{tree})}$ for external data in the other three regions. We will elaborate on this result in the conclusions presented in chapter 5. See also appendix B.2 where we investigate a parity symmetry of $\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ that relates two of the four regions in (4.170).

4.3.3.6 Four-Site Invariant for $\mathfrak{u}(2,2|0+4)$ and MHV Superamplitude

Let us continue with an example that yields the deformation $\mathcal{A}_{4,2}^{(\text{def.})}$ of the $\mathcal{N}=4$ SYM MHV superamplitude $\mathcal{A}_{4,2}^{(\text{tree})}$. To this end, we evaluate the unitary Graßmannian integral

(4.111) for the invariant with $(N,K)=(4,2)$ of the superalgebra $\mathfrak{u}(2,2|0+4)$. We proceed analogously to the bosonic case presented in section 4.3.3.4. Instead of (4.145), we end up with

$$\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}},\boldsymbol{\eta})=8i\,\delta^{4|0}(P)\mathscr{F}(\mathcal{C})\delta^{0|8}(\boldsymbol{\eta}^{\mathrm{d}}-\overline{\mathcal{C}}\boldsymbol{\eta}^{\mathrm{o}})\,. \tag{4.173}$$

Here the momentum conserving delta function can be found in (4.116) and the matrix $\mathcal{C}$ is given in (4.153) in terms of the external data $\boldsymbol{\lambda}$. The function $\mathscr{F}$ is specified in (4.32). Note that the exponents in this function depend on the algebra and thus differ slightly from the bosonic case. We obtain from (4.173) the final expression

$$\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}},\boldsymbol{\eta})=8i\,\frac{\delta^{4|0}(P)\delta^{0|8}(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(\frac{\langle 14\rangle}{\langle 34\rangle}\right)^{c_1}\left(\frac{\langle 12\rangle}{\langle 14\rangle}\right)^{c_2}\left(\frac{\langle 34\rangle\overline{\langle 34\rangle}}{\langle 14\rangle\overline{\langle 14\rangle}}\right)^{v_1-v_2} \tag{4.174}$$

with the supermomentum conserving delta function (4.117). Employing the relation (4.155) for the fraction in the last bracket, we realize that $\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}},\boldsymbol{\eta})$ is proportional to the deformed MHV superamplitude $\mathcal{A}_{4,2}^{(\text{def.})}$ from (1.48).

At this point we pause for a moment to bring to mind a structural connection with results discussed earlier in this thesis. In section 4.1.5.2 we obtained the Yangian invariant $|\Psi_{4,2}\rangle$ for oscillator representations of $\mathfrak{u}(p,q|r+s)$ from a unitary Graßmannian integral. Already prior to this, in section 2.4.2.3 we pointed out that $|\Psi_{4,2}\rangle$ for $\mathfrak{u}(2,2|4)$ is essentially the R-matrix of the planar $\mathcal{N}=4$ SYM one-loop spectral problem. The function $\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}},\boldsymbol{\eta})$ computed in this section, and thus the deformed amplitude $\mathcal{A}_{4,2}^{(\text{def.})}$, is the very same Yangian invariant as $|\Psi_{4,2}\rangle$. It is merely expressed in a different basis, i.e. spinor helicity variables instead of oscillators. The idea to identify the R-matrix of the spectral problem with a deformation of the amplitude $\mathcal{A}_{4,2}^{(\text{tree})}$ goes back to [103] and was in part inspired by [102]. However, the necessary change of basis has never been worked out explicitly. We filled this gap with the Bargmann transformation of section 4.2.[4]

4.3.3.7 Six-Site Invariant for $\mathfrak{u}(2,2|0+4)$ and NMHV Superamplitude

Here we extend the calculation of the six-site Yangian invariant for $\mathfrak{u}(2,2)$ from section 4.3.3.5 to the superalgebra $\mathfrak{u}(2,2|0+4)$. We compare our result to the NMHV superamplitude $\mathcal{A}_{6,3}^{(\text{tree})}$ of $\mathcal{N}=4$ SYM. Furthermore, we comment on its relation to the deformed superamplitude $\mathcal{A}_{6,3}^{(\text{def.})}$ whose existence has not been settled yet.

Computing the unitary Graßmannian integral (4.111) for $(N,K)=(6,3)$ and the superalgebra $\mathfrak{u}(2,2|0+4)$, the bosonic equation (4.159) gets replaced by

$$\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}},\boldsymbol{\eta})=32\frac{\delta^{4|0}(P)}{s_{123}}\int_0^{2\pi}\mathrm{d}\gamma\,\mathscr{F}(\mathcal{C}(\gamma))\delta^{0|12}(\boldsymbol{\eta}^{\mathrm{d}}-\overline{\mathcal{C}(\gamma)}\boldsymbol{\eta}^{\mathrm{o}}) \tag{4.175}$$

with $\mathcal{C}(\gamma)$ given in terms of the external data $\boldsymbol{\lambda}$ by (4.164). Manipulating the fermionic delta function and using the form of the function $\mathscr{F}$ in (4.33) leads to

$$\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}},\boldsymbol{\eta})=32\frac{\delta^{4|0}(P)\delta^{0|8}(Q)}{s_{123}^5}\mathcal{I}(v_1-v_2,v_2-v_3,c_1,c_2,c_3)\,, \tag{4.176}$$

[4] In the last paragraph of section 2.4.2.3 we brought to attention that slightly non-standard "oscillator" representations are frequently used in the $\mathcal{N}=4$ SYM spectral problem. This subtlety is not taken into account by our transformation.

where

$$
\begin{aligned}
\mathcal{I}(z_1,z_2,c_1,c_2,c_3) = \int_0^{2\pi} \mathrm{d}\gamma \, & \frac{\delta^{0|4}(\mathtt{a}+e^{-i\gamma}\mathtt{b})}{(-e^{i\gamma})^{-2-c_2}|\mathtt{A}-e^{i\gamma}\mathtt{B}|^{2(1+z_1)}(\mathtt{A}-e^{i\gamma}\mathtt{B})^{c_2-c_1}} \\
& \cdot \frac{1}{|\mathtt{C}-e^{-i\gamma}\mathtt{D}|^{2(1+z_2)}(\mathtt{C}-e^{-i\gamma}\mathtt{D})^{c_3-c_2}} \, .
\end{aligned}
\tag{4.177}
$$

Here we shifted the integration variable γ as in the bosonic case and the quantities $\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{D}$ are defined in (4.165). Furthermore, we introduced the Graßmann variables

$$
\mathtt{a} = \overline{\langle 23\rangle}\eta^1 + \overline{\langle 31\rangle}\eta^2 + \overline{\langle 12\rangle}\eta^3 \, , \quad \mathtt{b} = \overline{\langle 56\rangle}\eta^4 + \overline{\langle 64\rangle}\eta^5 + \overline{\langle 45\rangle}\eta^6
\tag{4.178}
$$

with the vectors $\eta^i = (\eta^i_{\dot a})$. The expression in (4.176) with the integral (4.177) is our ultimate result for the Yangian invariant $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}},\boldsymbol{\eta})$ with generic deformation parameters $v_1,v_2,v_3\in\mathbb{C}$ and $c_1,c_2,c_3\in\mathbb{Z}$. It should be thought of as the deformed superamplitude $\mathcal{A}_{6,3}^{(\text{def.})}$ whose existence has been questioned, see section 1.3.6.

However, this interpretation of $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}},\boldsymbol{\eta})$ comes with a caveat. As for the bosonic version, its undeformed limit reduces only in a certain kinematic region to the superamplitude $\mathcal{A}_{6,3}^{(\text{tree})}$. Let us show this explicitly. For this purpose we study the integral (4.177) in the special case $z_1=z_2=0$ and $c_1=c_2=c_3$. Introducing $u=e^{i\gamma}$, it becomes

$$
\begin{aligned}
\mathcal{I}(0,0,c_1,c_1,c_1) &= \frac{i}{\overline{\mathtt{A}}\mathtt{BC}\overline{\mathtt{D}}} \oint \mathrm{d}u \, \frac{\delta^{0|4}(u\mathtt{a}+\mathtt{b})}{(-u)^{1-c_1}\Big(u-\frac{\mathtt{A}}{\mathtt{B}}\Big)\Big(u-\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}\Big)\Big(u-\frac{\mathtt{D}}{\mathtt{C}}\Big)\Big(u-\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}\Big)} \\
&= \oint \mathrm{d}u \, \mathscr{I}(u) \, ,
\end{aligned}
\tag{4.179}
$$

where we integrate counterclockwise along the unit circle. We denote the integrand by $\mathscr{I}(u)$. Let us compute its residues,

$$
\operatorname{res}_0 \mathscr{I}(u) = \begin{cases}
\begin{aligned}
-i\frac{\delta^{0|4}(\mathtt{b})}{\mathtt{A}\overline{\mathtt{BC}}\mathtt{D}} &= -i\frac{s_{123}^4\delta^{0|4}(\overline{\langle 56\rangle}\eta^4+\overline{\langle 64\rangle}\eta^5+\overline{\langle 45\rangle}\eta^6)}{\langle 1|2+3|\overline{4\rangle}\,\overline{\langle 56\rangle}\langle 23\rangle\langle 3|1+2|\overline{6\rangle}\,\overline{\langle 45\rangle}\langle 12\rangle} \\
&= -i\frac{s_{123}^4\delta^{0|4}([65]\eta^4+[46]\eta^5+[54]\eta^6)}{\langle 1|2+3|4][65]\langle 23\rangle\langle 3|1+2|6][54]\langle 12\rangle}
\end{aligned} & \text{for} \quad c_1 = 0 \, , \\
0 & \text{for} \quad c_1 > 0 \, ,
\end{cases}
$$

$$
\operatorname{res}_\infty \mathscr{I}(u) = \begin{cases}
\begin{aligned}
i\frac{\delta^{0|4}(\mathtt{a})}{\overline{\mathtt{A}}\mathtt{BC}\overline{\mathtt{D}}} &= i\frac{s_{123}^4\delta^{0|4}(\overline{\langle 23\rangle}\eta^1+\overline{\langle 31\rangle}\eta^2+\overline{\langle 12\rangle}\eta^3)}{\overline{\langle 1|}2+3|4\rangle\langle 56\rangle\overline{\langle 23\rangle}\,\overline{\langle 3|}1+2|6\rangle\langle 45\rangle\overline{\langle 12\rangle}} \\
&= i\frac{s_{123}^4\delta^{0|4}([32]\eta^1+[13]\eta^2+[21]\eta^3)}{[1|2+3|4\rangle\langle 56\rangle[32][3|1+2|6\rangle\langle 45\rangle[21]}
\end{aligned} & \text{for} \quad c_1 = 0 \, , \\
0 & \text{for} \quad c_1 < 0 \, ,
\end{cases}
$$

$$
\begin{aligned}
\operatorname{res}_{\frac{\mathtt{A}}{\mathtt{B}}} \mathscr{I}(u) &= -i\frac{\big(-\frac{\mathtt{A}}{\mathtt{B}}\big)^{c_1}\,\delta^{0|4}(\mathtt{Aa}+\mathtt{Bb})}{\mathtt{AB}(|\mathtt{A}|^2-|\mathtt{B}|^2)(\mathtt{AC}-\mathtt{BD})(\mathtt{A}\overline{\mathtt{D}}-\mathtt{B}\overline{\mathtt{C}})} \\
&= -i\frac{s_{123}^5}{\langle 5|1-6|\overline{2\rangle}\langle 16\rangle\overline{\langle 34\rangle}}\,\frac{\delta^{0|4}(\overline{\langle 43\rangle}\eta^2+\overline{\langle 24\rangle}\eta^3+\overline{\langle 23\rangle}\eta^4)}{s_{234}\langle 1|2+3|\overline{4\rangle}\langle 56\rangle\overline{\langle 23\rangle}}\left(-\frac{\langle 1|2+3|\overline{4\rangle}}{\langle 56\rangle\overline{\langle 23\rangle}}\right)^{c_1} \\
&= i\frac{s_{123}^5}{\langle 5|1+6|2]\langle 16\rangle[34]}\,\frac{\delta^{0|4}([43]\eta^2+[24]\eta^3+[32]\eta^4)}{s_{234}\langle 1|2+3|4]\langle 56\rangle[32]}\left(-\frac{\langle 1|2+3|4]}{\langle 56\rangle[32]}\right)^{c_1} \, ,
\end{aligned}
$$

$$
\operatorname{res}_{\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}} \mathscr{I}(u) = i\frac{\Big(-\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}\Big)^{c_1}\,\delta^{0|4}(\overline{\mathtt{B}}\mathtt{a}+\overline{\mathtt{A}}\mathtt{b})}{\overline{\mathtt{AB}}(|\mathtt{A}|^2-|\mathtt{B}|^2)(\overline{\mathtt{AC}}-\overline{\mathtt{BD}})(\overline{\mathtt{A}}\mathtt{D}-\overline{\mathtt{B}}\mathtt{C})}
$$

$$
\begin{aligned}
&= \quad i\frac{s_{123}^5}{\overline{\langle 5|}1-6|2\rangle\overline{\langle 16\rangle}\langle 34\rangle}\frac{\delta^{0|4}(\overline{\langle 56\rangle}\eta^1+\overline{\langle 16\rangle}\eta^5+\overline{\langle 51\rangle}\eta^6)}{s_{234}\overline{\langle 1|}2+3|4\rangle\overline{\langle 56\rangle}\langle 23\rangle}\left(-\frac{\overline{\langle 56\rangle}\langle 23\rangle}{\overline{\langle 1|}2+3|4\rangle}\right)^{c_1}\\
&= -i\frac{s_{123}^5}{[5|1+6|2\rangle[16]\langle 34\rangle}\frac{\delta^{0|4}([65]\eta^1+[16]\eta^5+[51]\eta^6)}{s_{234}[1|2+3|4\rangle[65]\langle 23\rangle}\left(\frac{[65]\langle 23\rangle}{[1|2+3|4\rangle}\right)^{c_1},\\
\operatorname{res}_{\frac{\mathtt{D}}{\mathtt{C}}}\mathscr{I}(u) &= -i\frac{\left(-\frac{\mathtt{D}}{\mathtt{C}}\right)^{c_1}\delta^{0|4}(\mathtt{Da}+\mathtt{Cb})}{\mathtt{CD}(|\mathtt{C}|^2-|\mathtt{D}|^2)(\mathtt{AC}-\mathtt{BD})(\overline{\mathtt{A}}\mathtt{D}-\overline{\mathtt{B}}\mathtt{C})}\\
&= -i\frac{s_{123}^5}{\overline{\langle 5|}1-6|2\rangle\langle 16\rangle\overline{\langle 34\rangle}}\frac{\delta^{0|4}(\overline{\langle 45\rangle}\eta^3+\overline{\langle 35\rangle}\eta^4+\overline{\langle 43\rangle}\eta^5)}{s_{126}\overline{\langle 3|}1+2|6\rangle\overline{\langle 45\rangle}\langle 12\rangle}\left(-\frac{\overline{\langle 45\rangle}\langle 12\rangle}{\overline{\langle 3|}1+2|6\rangle}\right)^{c_1}\\
&= \quad i\frac{s_{123}^5}{[5|1+6|2\rangle\langle 16\rangle[34]}\frac{\delta^{0|4}([54]\eta^3+[35]\eta^4+[43]\eta^5)}{s_{126}[3|1+2|6\rangle[54]\langle 12\rangle}\left(\frac{[54]\langle 12\rangle}{[3|1+2|6\rangle}\right)^{c_1},\\
\operatorname{res}_{\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}}\mathscr{I}(u) &= \quad i\frac{\left(-\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}\right)^{c_1}\delta^{0|4}(\overline{\mathtt{C}}\mathtt{a}+\overline{\mathtt{D}}\mathtt{b})}{\overline{\mathtt{CD}}(|\mathtt{C}|^2-|\mathtt{D}|^2)(\overline{\mathtt{AC}}-\overline{\mathtt{BD}})(\mathtt{A}\overline{\mathtt{D}}-\mathtt{B}\overline{\mathtt{C}})}\\
&= \quad i\frac{s_{123}^5}{\langle 5|1-6\overline{|2\rangle}\,\overline{\langle 16\rangle}\langle 34\rangle}\frac{\delta^{0|4}(\overline{\langle 62\rangle}\eta^1+\overline{\langle 16\rangle}\eta^2+\overline{\langle 12\rangle}\eta^6)}{s_{126}\langle 3|1+2\overline{|6\rangle}\langle 45\rangle\overline{\langle 12\rangle}}\left(-\frac{\langle 3|1+2\overline{|6\langle}}{\langle 45\rangle\overline{\langle 12\rangle}}\right)^{c_1}\\
&= -i\frac{s_{123}^5}{\langle 5|1+6|2][16]\langle 34\rangle}\frac{\delta^{0|4}([62]\eta^1+[16]\eta^2+[21]\eta^6)}{s_{126}\langle 3|1+2|6]\langle 45\rangle[21]}\left(-\frac{\langle 3|1+2|6]}{\langle 45\rangle[21]}\right)^{c_1}.
\end{aligned}
\tag{4.180}
$$

We used the momentum and supermomentum conservation in (4.176) to obtain these formulas. What is more, we presented three expressions for each residue. The first two make the analytic structure most transparent. The third version involving the angle brackets is obtained using (4.150). It is appropriate to identify the residues with known expressions later on. By means of the residue theorem, the integral (4.177) then becomes

$$
\mathcal{I}(0,0,c_1,c_1,c_1) = 2\pi i\begin{cases}
\operatorname{res}_0\mathscr{I}(u)+\operatorname{res}_{\frac{\mathtt{A}}{\mathtt{B}}}\mathscr{I}(u)+\operatorname{res}_{\frac{\mathtt{D}}{\mathtt{C}}}\mathscr{I}(u) & \text{for} \quad \begin{matrix} s_{234}>0\,, \\ s_{126}<0\,, \end{matrix}\\
\operatorname{res}_0\mathscr{I}(u)+\operatorname{res}_{\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}}\mathscr{I}(u)+\operatorname{res}_{\frac{\mathtt{D}}{\mathtt{C}}}\mathscr{I}(u) & \text{for} \quad \begin{matrix} s_{234}<0\,, \\ s_{126}<0\,, \end{matrix}\\
\operatorname{res}_0\mathscr{I}(u)+\operatorname{res}_{\frac{\mathtt{A}}{\mathtt{B}}}\mathscr{I}(u)+\operatorname{res}_{\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}}\mathscr{I}(u) & \text{for} \quad \begin{matrix} s_{234}>0\,, \\ s_{126}>0\,, \end{matrix}\\
\operatorname{res}_0\mathscr{I}(u)+\operatorname{res}_{\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}}\mathscr{I}(u)+\operatorname{res}_{\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}}\mathscr{I}(u) & \text{for} \quad \begin{matrix} s_{234}<0\,, \\ s_{126}>0\,. \end{matrix}
\end{cases}
\tag{4.181}
$$

The four cases, which are distinguished by kinematic regions of the external data $\boldsymbol{\lambda}$, appear in precisely the same manner as for the bosonic version in (4.170). At this point we insert a remark. Because the sum of all residues vanishes, we can express (4.181) for all $c_1 \in \mathbb{Z}$ in terms of the residues computed in (4.180). That is even though we evaluated $\operatorname{res}_0\mathscr{I}(u)$ and $\operatorname{res}_\infty\mathscr{I}(u)$ only for restricted ranges of c_1.

Finally, we want to compare our findings with the superamplitude $\mathcal{A}_{6,3}^{(\text{tree})}$ of $\mathcal{N}=4$ SYM presented in (1.26). Thus we have to specialize to the representation label $c_1 = 0$, see the discussion after (4.105). In this case the residues can be expressed in terms of the quantities $\mathcal{R}^{r;st}$ from (1.28),

$$
\begin{aligned}
\operatorname{res}_0 \mathscr{I}(u)\Big|_{c_1=0} &= -i \frac{s_{123}^5 \mathcal{R}^{1;46}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\,, \\
\operatorname{res}_\infty \mathscr{I}(u)\Big|_{c_1=0} &= i \frac{s_{123}^5 \mathcal{R}^{6;24}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\,, \\
\operatorname{res}_{\frac{\mathtt{A}}{\mathtt{B}}} \mathscr{I}(u)\Big|_{c_1=0} &= -i \frac{s_{123}^5 \mathcal{R}^{1;35}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\,, \\
\operatorname{res}_{\frac{\bar{\mathtt{B}}}{\bar{\mathtt{A}}}} \mathscr{I}(u)\Big|_{c_1=0} &= i \frac{s_{123}^5 \mathcal{R}^{6;25}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\,, \\
\operatorname{res}_{\frac{\mathtt{D}}{\mathtt{C}}} \mathscr{I}(u)\Big|_{c_1=0} &= i \frac{s_{123}^5 \mathcal{R}^{6;35}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\,, \\
\operatorname{res}_{\frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}} \mathscr{I}(u)\Big|_{c_1=0} &= -i \frac{s_{123}^5 \mathcal{R}^{1;36}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\,,
\end{aligned}
\tag{4.182}
$$

where we used once more momentum and supermomentum conservation. The statement that the sum over all residues vanishes translates into the identity

$$
\mathcal{R}^{6;24} + \mathcal{R}^{6;25} + \mathcal{R}^{6;35} = \mathcal{R}^{1;35} + \mathcal{R}^{1;36} + \mathcal{R}^{1;46}\,,
\tag{4.183}
$$

cf. equation (4.20) in [79]. The invariant (4.176) with (4.181) in the region $s_{234}, s_{126} > 0$ and for $c_1 = 0$ becomes

$$
\begin{aligned}
\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}, \boldsymbol{\eta}) &= 64\pi i \delta^{4|0}(P)\delta^{0|8}(Q) \frac{\operatorname{res}_0 \mathscr{I}(u) + \operatorname{res}_{\frac{\mathtt{A}}{\mathtt{B}}} \mathscr{I}(u) + \operatorname{res}_{\frac{\bar{\mathtt{C}}}{\bar{\mathtt{D}}}} \mathscr{I}(u)}{s_{123}^5} \\
&= 64\pi \delta^{4|0}(P)\delta^{0|8}(Q) \frac{\mathcal{R}^{1;46} + \mathcal{R}^{1;35} + \mathcal{R}^{1;36}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\,.
\end{aligned}
\tag{4.184}
$$

Up to a numerical prefactor, this is $\mathcal{A}_{6,3}^{(\text{tree})}$ from (1.26). Thus the situation for the $\mathfrak{u}(2,2|0+4)$ Yangian invariant $\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}, \boldsymbol{\eta})$, which we obtained from the unitary Graßmannian integral (4.111), is completely analogous to the bosonic case of section 4.3.3.5. In one of the four kinematic regions it agrees with the amplitude and in the other three it does not seem to match. Once again we refer the reader to the conclusions in chapter 5 for further thoughts on this issue.

Chapter 5

Conclusions and Outlook

In this dissertation we investigated the integrable structure of tree-level $\mathcal{N} = 4$ SYM scattering amplitudes. It manifests itself in the invariance of these amplitudes under the Yangian of the superconformal algebra $\mathfrak{psu}(2,2|4)$. We adopted a mathematical perspective on this topic by studying Yangian invariants of the large class of algebras $\mathfrak{u}(p,q|m)$. This broadened scope revealed fascinating connections to various concepts in the field of integrable models. Notably, in chapter 2 we derived an elegant characterization of Yangian invariants within the QISM, which is an extensive toolbox to study integrable spin chains. It allowed us to show that Yangian invariants for $\mathfrak{u}(2)$ can be constructed by means of a Bethe ansatz in chapter 3. Therein we also established a link to vertex models of statistical physics. A complementary method for the construction of a class of Yangian invariants for $\mathfrak{u}(p,q|m)$ was developed in chapter 4. It has its origin in the Graßmannian integral formulation of scattering amplitudes and generalizes certain unitary matrix models. In the present chapter we recapitulate these main results and highlight further findings chapter by chapter. Along the way we provide some additional perspectives and mention future directions. In particular, we emphasize the relevance of our work for the "physical" problem we started out with, i.e. understanding the integrable structure of planar $\mathcal{N} = 4$ SYM amplitudes. Finally, we conclude with some general remarks.

The goal of **chapter 2** was to establish a common algebraic and representation theoretic language that covers integrable spin chains and amplitudes alike, and can be applied for our study of Yangian invariants throughout this thesis. The use of the QISM form of the Yangian of $\mathfrak{gl}(n|m)$ was instrumental for achieving this. In our review of this formulation in section 2.1 we emphasized that the Yangian generators can be obtained from an expansion of a spin chain monodromy in its spectral parameter. Building on this, we translated the Yangian invariance condition into the QISM language in section 2.2. We identified Yangian invariants with spin chain states that are specific eigenstates of the monodromy matrix elements. This allows to interpret the deformation parameters of the amplitudes in section 1.3.6 as inhomogeneities and representation labels of spin chains. Our formulation of the Yangian invariance condition makes the tools of the QISM applicable for the construction of such invariants. Some of these tools were put to use in chapter 3. The second cornerstone for achieving the aforementioned goal was the choice of certain oscillator representations of $\mathfrak{u}(p,q|m) \subset \mathfrak{gl}(n|m)$, which we reviewed in section 2.3, at the spin chain sites. These include a wide range of spin chain models. They allow to interpolate between the spin $\frac{1}{2}$ representation of $\mathfrak{su}(2)$, which appears in the Heisenberg model, and an infinite-dimensional representation of $\mathfrak{psu}(2,2|4)$ that features in $\mathcal{N} = 4$ SYM. We found that for the construction of Yangian invariants we actually need two series of oscillator representations, $\mathcal{D}_c$ and the dual family $\bar{\mathcal{D}}_c$. Importantly, the representation label c has to be an integer in order to stay inside the Fock space. Such constraints on the representation labels are usually neglected in the context of deformed amplitudes, cf. section 1.3.6. However, they proved to be crucial for the results we obtained in chapter 4. Furthermore, we paid attention to the conjugation properties of the oscillators building up the representations. These are of importance because they determine the generators (2.36) at the dual sites. In addition, they influence the real form $\mathfrak{u}(p,q|m)$ of $\mathfrak{gl}(p+q|m)$. The impact of this real form is nicely illustrated by the two-site sample invariant $|\Psi_{2,1}\rangle$ from section 2.4.2.1. In the compact case this invariant is just a monomial, while in the non-compact setting it is a Bessel function, which has an infinite power series expansion. Let us also recapitulate the four-site sample invariant $|\Psi_{4,2}\rangle$ from sections 2.4.1.5 and 2.4.2.3. Its Yangian invariance condition is equivalent to a Yang-Baxter equation. Hence $|\Psi_{4,2}\rangle$ can be interpreted as an R-matrix whose spectral parameter z is given by the difference of two inhomogeneities of the associated monodromy $M_{4,2}(u)$. The two "spectral parameters" z and u have to be distinguished. We also pointed out that for $\mathfrak{u}(2,2|4)$ this R-matrix is essentially that of the one-loop $\mathcal{N}=4$ SYM spin chain. This interesting link to the spectral problem was first observed in [103, 104].

Chapter 2 led to some open questions which deserve further attention. In section 2.4.2.2 we observed that non-compact three-site invariants do not seem to exist for all algebras $\mathfrak{u}(p,q|m)$. We constructed a sample invariant for $\mathfrak{u}(p,1)$ but we were not able to do so for $\mathfrak{u}(2,2)$, which is in agreement with the non-existence of three-particle scattering amplitudes for real momenta. It would be desirable to study systematically for which algebras there can be a Yangian invariant $|\Psi_{N,K}\rangle$. Because this invariant is an element of the tensor product of $N-K$ ordinary representations $\mathcal{D}_c$ and K dual ones $\bar{\mathcal{D}}_c$, this can be investigated by comparing the decomposition of the $N-K$-fold tensor product of $\mathcal{D}_c$ with that of the K-fold one. These decompositions were studied in [141] for bosonic algebras and in [153] for superalgebras. Another starting point for inquiries is formula (2.88) that expresses the compact Yangian invariant $|\Psi_{4,2}\rangle$ in terms of the Gauß hypergeometric function ${}_2F_1$. There is a class of multivariate hypergeometric functions defined on the complex Graßmannian $\mathrm{Gr}(N,K)$ which reduces to Gauß' function for $\mathrm{Gr}(4,2)$. This class is discussed in [205], see also the substantial review [206] and the lightning introduction [207] to similar functions. It remains to be seen whether the invariant $|\Psi_{N,K}\rangle$ can be related to said functions on $\mathrm{Gr}(N,K)$.[1] On a different note, it would be very illuminating to obtain a closed expression for the non-compact invariant $|\Psi_{4,2}\rangle$ in (2.99), which is presently only available in form of a multiple sum.

The major observation of **chapter 3** was that Yangian invariants are specific eigenstates of spin chain transfer matrices. This follows directly from our QISM characterization of Yangian invariants put forward in chapter 2. It applies to invariants with representations of the superalgebra[2] $\mathfrak{u}(p,q|m)$ and thus in particular to the deformed amplitudes of section 1.3.6 that are associated with the superconformal algebra $\mathfrak{psu}(2,2|4)$. The diagonalization of spin chain transfer matrices is the central problem addressed by the toolbox of the QISM. Therefore these tools should be applicable for the construction of Yangian invariants. In section 3.1 we reviewed the most basic method of the toolbox, i.e. the algebraic Bethe ansatz for the $\mathfrak{su}(2)$ Heisenberg spin chain. Thereby we filled in some details on this model, which already served as an example in the introductory section 1.1. In section 3.2 we employed this Bethe ansatz for the construction of $\mathfrak{u}(2)$ Yangian invariants, which can be considered as toy models for amplitudes. We showed that those invariants are special

[1] Relations between multivariate hypergeometric functions and Yangian invariants were observed independently in the context of deformed amplitudes [110].

[2] Note that we spelled out the argument explicitly only in the bosonic case in section 3.2.

Bethe vectors. The Bethe roots parameterizing these vectors are obtained from functional relations that are a special case of the usual Baxter equation. Remarkably, unlike the full equation, this special case can be solved with ease. The Bethe roots form real strings in the complex plane and also the admissible inhomogeneities and representation labels of the monodromy are constrained. What is more, we found a simple superposition principle for the solutions of the functional relations. While the simplest two-site Yangian invariant $|\Psi_{2,1}\rangle$ corresponds to one string of Bethe roots, the four-site invariant $|\Psi_{4,2}\rangle$ is obtained by placing two of those strings in the complex plane. Furthermore, our work on the Bethe ansatz led to a classification of $\mathfrak{u}(2)$ Yangian invariants in terms of permutations in [3]. Astonishingly, these permutations also appear in the study of deformed amplitudes, cf. section 1.3.6. This illustrates the importance of our Bethe ansatz for the investigation of the structure of Yangian invariants even far beyond the $\mathfrak{u}(2)$ case. Finally, in section 3.4 we identified Yangian invariants with partition functions of certain vertex models on in general non-rectangular lattices. In particular, we demonstrated that our Bethe ansatz for $\mathfrak{u}(2)$ Yangian invariants generalizes the rational limit of Baxter's perimeter Bethe ansatz, which only covers the spin $\frac{1}{2}$ representation.

Our work on the Bethe ansatz for Yangian invariants in **chapter 3** suggests several natural generalizations. The most urgent issue is to find an efficient technique to evaluate the Yangian invariant Bethe vectors. For the sample invariants in section 3.2.2 we computed the Bethe vectors explicitly for small representation labels to arrive at the formulas involving the oscillator contractions $(k \bullet l)$. We were able to surpass this method for the invariant $|\Psi_{2,1}\rangle$ by the elegant derivation presented in appendix A.1. Clearly it would be desirable to extend this method to further invariants, say $|\Psi_{4,2}\rangle$ to begin with. A different direction concerns the extension of our Bethe ansatz for $\mathfrak{u}(2)$ to more general algebras, i.e. the exploration of a larger part of the "landscape" of invariants in figure 1.2. There are no conceptual difficulties to cover the $\mathfrak{u}(n)$ case. In fact, we already provided the relevant functional relations in appendix A.2. On a technical level, however, the evaluation of the Bethe vectors appears to be quite intricate due to nesting. Yet our results in appendix A.3 show that at least for certain sample invariants the nesting completely disappears and thereby the complexity is reduced to that of the $\mathfrak{u}(2)$ case. These simplifications deserve continuing attention. Furthermore, the extension of the Bethe ansatz for Yangian invariants to compact superalgebras $\mathfrak{u}(n|m)$, e.g. along the lines of [124], should impose no obstacles. Interesting conceptual questions are to be expected in the non-compact $\mathfrak{u}(p,q|m)$ setting. The algebraic Bethe ansatz for $\mathfrak{u}(2)$ is based on a reference state $|\Omega\rangle$, which is given in (3.17) by the tensor product of the highest weight states at the spin chain sites. However, out of the non-compact oscillator representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_c$ only the latter has a highest weight, cf. section 2.3. Thus there is no such reference state in the non-compact case. A way out of this dilemma might be to replace the algebraic Bethe ansatz with another method from the QISM toolbox. A suitable tool might be Sklyanin's separation of variables [123, 208] as it does not require a reference state. Some recent developments of this method are discussed e.g. in [209]. Let us remark that it was also used to solve the $\mathfrak{sl}(\mathbb{C}^2)$ spin chain appearing in QCD [23], which we encountered as an example in section 1.1. Lastly, we want to mention work that potentially has close ties with our Bethe ansatz for Yangian invariants. In [210] a set of equations characterizing form factors of $1+1$-dimensional integrable quantum field theories was solved in terms of Bethe vectors. One of these equations appears to generalize our QISM version (2.24) of the Yangian invariance condition. This equation is also known to be related to a special case of the so-called quantum Knizhnik-Zamolodchikov (qKZ) equation [211], which is solved by means of Bethe ansatz techniques in [212].[3]

[3]Similarities between the Yangian invariance condition and the qKZ equation were pointed out indepen-

In **chapter 4** we advocated to equip the Graßmannian integral with a unitary contour. We were led to that idea by applying this method from the field of $\mathcal{N}=4$ SYM scattering amplitudes, cf. sections 1.3.5 and 1.3.6, to the construction of Yangian invariants of a wide range of algebras. We started out in section 4.1 by utilizing the Graßmannian integral to build Yangian invariants $|\Psi_{N=2K,K}\rangle$ with oscillator presentations of $\mathfrak{u}(p,q|m)$. In the resulting formula the usual formal delta functions of spinor helicity variables of the SYM case in section 1.3.5 are replaced by an exponential function of oscillators. We were able to choose the integration variable $\mathcal{C}$ to be the unitary group manifold $U(K)$. We found that this contour eliminates branch cuts of the integrand which plagued the Graßmannian integral for deformed amplitudes in section 1.3.6. This observation is tightly interlocked with the restriction to integer representation labels c_i. Let us emphasize that the choice of the contour is independent of the algebra. Notably, it works for compact and non-compact algebras alike. We termed our construction unitary Graßmannian matrix model because for special values of the deformation parameters v_i, c_i it reduces to a well-known unitary matrix model, the Brezin-Gross-Witten model. Our reasoning implies that this matrix model is Yangian invariant in the external source fields. We evaluated our unitary Graßmannian matrix model for several sample invariants, some which were obtained by other means in chapters 2 and 3. In particular, we evaluated the invariant $|\Psi_{4,2}\rangle$, i.e. the R-matrix, which takes the form of a $U(2)$ matrix integral in our approach. This formula for the R-matrix is an example of ideas from scattering amplitudes contributing to our understanding of integrable models as such. In section 4.2 we established a direct connection between our results for oscillator representations of $\mathfrak{u}(p,q=p|m)$ and the study of deformed scattering amplitudes by applying a change of basis to spinor helicity-like variables. Technically, this was implemented by a Bargmann transformation, which is an integral transformation known from the harmonic oscillator in quantum mechanics. We identified those oscillator representations that are relevant for tree-level amplitudes by matching the $\mathfrak{u}(2,2|4)$ symmetry generators. The "ordinary" representation $\mathcal{D}_0$ corresponds to a particle with positive energy, which is enforced by $(\tilde{\lambda}_{\dot{\alpha}}) = +(\overline{\lambda}_\alpha)$. Likewise, the dual representation $\bar{\mathcal{D}}_0$ is associated with a negative energy particle, i.e. $(\tilde{\lambda}_{\dot{\alpha}}) = -(\overline{\lambda}_\alpha)$. Computing the Bargmann transformation of the unitary Graßmannian matrix model for $\mathfrak{u}(p,p|m)$ yielded a formula for the Yangian invariant $\Psi_{2K,K}$ in spinor helicity-like variables that we investigated in section 4.3. It refines the Graßmannian formula for deformed amplitudes of section 1.3.6 in several aspects. Importantly, it is defined for the physical Minkowski signature instead of split signature or complexified momentum space. We showed that for this signature and our order of positive and negative energy particles the choice of the unitary contour is, in a way, dictated by momentum conservation. Lastly, we put our formula to the test by evaluating sample invariants in the $\mathfrak{u}(2,2|4)$ case. We identified $\Psi_{4,2}$ with the deformed amplitude $\mathcal{A}_{4,2}^{(\text{def.})}$. Thereby we constructed is essence an explicit change of basis from the oscillator R-matrix of the planar $\mathcal{N}=4$ SYM one-loop spectral problem to this deformed amplitude. Furthermore, we evaluated the invariant $\Psi_{6,3}$, which is a natural candidate for the presently unknown deformed amplitude $\mathcal{A}_{6,3}^{(\text{def.})}$. Vexingly, however, its undeformed limit $v_i, c_i = 0$ coincides with the amplitude $\mathcal{A}_{6,3}^{(\text{tree})}$ merely in the kinematic region $s_{234}, s_{126} > 0$.

Obviously, the most pressing open problem in **chapter 4** is to clarify the precise relation between the unitary Graßmannian matrix integral for the undeformed $\Psi_{6,3}$ in the $\mathfrak{u}(2,2|4)$ case and the NMHV superamplitude $\mathcal{A}_{6,3}^{(\text{tree})}$. Let us elaborate on some of the logical possibilities for the apparent mismatch between these two quantities. First, one might

dently in [110].

question the correctness of our evaluation of the unitary Graßmannian integral (4.111) for the $\mathfrak{u}(2,2)$ version of $\Psi_{6,3}$ in section 4.3.3.5. Here we restrict to the bosonic setting because supersymmetry does not seem to affect the essential features. We certainly did make systematic mistakes because we only focused on the contribution with maximal kinematic support, which is proportional to a momentum conserving delta function. For example we explicitly assumed $\|\boldsymbol{\lambda}_1^{\mathrm{d}}\| \neq 0$ to obtain the intermediate step (4.157). Thus there might be additional terms in case this norm vanishes. However, such terms cannot make up for the mismatch between the undeformed Yangian invariant $\Psi_{6,3}$ and the gluon amplitude $A_{6,3}^{(\text{tree})}$ because of their restricted kinematic support. The most puzzling feature of the undeformed invariant $\Psi_{6,3}$ is the appearance of four kinematic regions in (4.170), in only one of which the invariant matches $A_{6,3}^{(\text{tree})}$. It would be highly desirable to have a principle argument for the necessity of these regions rather than merely the direct computation that led to (4.170). We addressed this point in appendix B.2, where we exposed a discrete parity symmetry of the unitary Graßmannian integral. We showed that the two kinematic regions of the simpler invariant $\Psi_{4,2}$ for $\mathfrak{u}(1,1)$ in (4.142) are inevitable because of this symmetry. In case of the $\mathfrak{u}(2,2)$ version of $\Psi_{6,3}$ the parity symmetry interrelates only two of the four regions in (4.170). In particular, the amplitude region $s_{234}, s_{126} > 0$ is parity invariant by itself. Therefore this symmetry of not sufficient to explain the need for all four regions. It would be interesting to search for a larger discrete symmetry group of $\Psi_{6,3}$ that does relate all regions. For this purpose it might be helpful to study in detail the kinematics of six massless relativistic particles e.g. by means of the techniques in [213]. Let us mention in this regard that we verified numerically the existence of physical momentum configurations in each of the four regions using [214]. The second logical possibility we want to comment on is that the unitary contour of the Graßmannian integral might be conceptually wrong for the construction of the sought after Yangian invariants. However, we provided a formal proof of the Yangian invariance of the Graßmannian integral with oscillator variables in section 4.1.2. Assumptions in this proof about the then unspecified contour were precisely satisfied by the selection of the unitary contour in section 4.1.4.1. What is more, we explicitly verified the Yangian symmetry of sample invariants computed using the unitary contour in section 4.1.5. Therefore we do not doubt the Yangian invariance of the unitary Graßmannian integral (4.24) in oscillator variables. We also have no reason to assume that this is affected by the Bargmann transformation to the spinor helicity version (4.111) of that integral. Still, one might argue that some amplitudes belong to a different class of Yangian invariants which is not captured by the unitary contour. We tried to address this concern in appendix B.1, where we showed in the bosonic case that once the $U(2)$ contour of $|\Psi_{4,2}\rangle$ is fixed, the $U(3)$ contour of $|\Psi_{6,3}\rangle$ follows from gluing. Recall in this context from section 4.3.3.4 the agreement between the Bargmann transformation of the undeformed $|\Psi_{4,2}\rangle$ for $\mathfrak{u}(2,2)$ and the gluon amplitude $A_{4,2}^{(\text{tree})}$. As a third logical possibility for the mismatch between the undeformed Yangian invariant $\Psi_{6,3}$ for $\mathfrak{u}(2,2|4)$ and the superamplitude $\mathcal{A}_{6,3}^{(\text{tree})}$, let us speculate that the symmetries of this amplitude might not quite be as commonly expected. The breakdown of its superconformal and Yangian invariance at certain singularities is well established, cf. [89, 90, 91]. Possibly these symmetries are in addition broken in a subtle way by the lack of the different kinematic regions, which are separated by such singularities. This speculation is motivated by our analysis of $\Psi_{4,2}$ for $\mathfrak{u}(1,1)$ in appendix B.2.2.2, where we discussed for this sample invariant the possibility of extending the expression from one of the two kinematic regions to the entire domain. Even though this function would satisfy the Yangian invariance condition (2.27) as a differential equation for generic λ_α^i, it would violate the parity symmetry. This symmetry is present in the oscillator invariant $|\Psi_{4,2}\rangle$ for $\mathfrak{u}(1,1)$ and therefore also in the associated R-matrix, see appendix B.2.2.1. The lesson learned from this example might apply to Yangian invariants with infinite-dimensional representations of $\mathfrak{u}(p,p|m)$ in general. In the spinor helicity-like basis it seems to be insufficient to verify the Yangian invariance condition (2.27) only for generic external data. In principle, one would probably also have to do the intricate analysis for external data at the singularities. In practice, however, it might often be possible to avoid this by exploiting discrete symmetries. This brings us back to the search for the discrete symmetry group of $\Psi_{6,3}$, which we already proposed above. On a slightly different note, we would like to mention the study of NMHV superamplitudes for $(2,2)$ signature in [215]. There the authors modified the usual expressions for these amplitudes by introducing sign factors that depend on certain kinematic regions. Only after this modification they were able to apply the conformal inversion, which is not an infinitesimal but a finite conformal transformation, to those expressions. Their kinematic regions are somewhat reminiscent of our regions for the $\mathfrak{u}(2,2)$ version of $\Psi_{6,3}$ in Minkowski signature in (4.170). Thus it might be instructive to compare the behavior of $\Psi_{6,3}$ and $A_{6,3}^{(\text{tree})}$ under finite conformal transformations, whose action is given e.g. in [199, 198, 200]. In this context it could also be necessary to think about "finite Yangian transformations". Ultimately, we could not clarify the relation between our function $\Psi_{6,3}$ for $\mathfrak{u}(2,2|4)$ and the NMHV superamplitude $\mathcal{A}_{6,3}^{(\text{tree})}$ in this thesis. We believe, however, that this issue is of crucial importance to understand the role of integrability for amplitudes.

Once this conceptual problem is resolved, **chapter 4** offers a host of interesting further directions to be explored. Clearly, our aim is to relate the Yangian invariant $\Psi_{2K,K}$ computed by the unitary Graßmannian integral (4.111) to the tree-level amplitude $\mathcal{A}_{2K,K}^{(\text{tree})}$ and deformations thereof,[4] at least in one kinematic region. This would demonstrate the relevance of the very natural unitary contour for a large class of tree-level amplitudes. Fortunately, this class contains amplitudes of all MHV degrees and thus ranges from amplitudes whose explicit expressions are very simple to those of immense complexity, cf. section 1.3.2. For the invariant $\Psi_{6,3}$ of $\mathfrak{u}(2,2|4)$ we reduced the defining $U(3)$ integral in (4.111) to a $U(1)$ integral that was then solved in the undeformed case by means of the residue theorem. The $U(K)$ integral for the invariant $\Psi_{2K,K}$ of $\mathfrak{u}(p,p|m)$ is expected to reduce to a $U(K-p)$ integral after exploiting the bosonic delta functions. In general, this is still a multi-dimensional integral, albeit with a fully specified contour. A technique to evaluate its undeformed version might be the global residue theorem, which has already been employed in the context of amplitudes [92], cf. section 1.3.5. Another puzzling question is to understand the geometric role of the $U(K)$ contour for $\mathcal{C}$ within the Graßmannian $\mathrm{Gr}(2K,K)$, that we parameterized in (1.41) by $C = \begin{pmatrix} 1_K & \vdots & \mathcal{C} \end{pmatrix}$. It would be desirable to specify the contour in a way that does not rely on this particular gauge fixing of C. Throughout chapter 4 we concentrated on invariants with $N = 2K$ because only in this case $\mathcal{C}$ is a *square* matrix. Naturally, we would like to extend the unitary Graßmannian integral (4.111) to $N \neq 2K$ and thereby access all tree-level amplitudes. Here the issue is to use an appropriate measure on the complex Stiefel manifold of *rectangular* $K \times (N-K)$ matrices $\mathcal{C}$ with $\mathcal{C}\mathcal{C}^\dagger = 1_{K\times K}$, see e.g. [216]. This generalizes the unitary group manifold to the case of rectangular matrices. The extension to $N \neq 2K$ is also of interest for the Graßmannian matrix model (4.24) for $|\Psi_{N,K}\rangle$ in the oscillator basis. In section 2.4.2.2 we observed that the $\mathfrak{gl}(p|r)$ invariant oscillator contractions $(k \bullet l)$ and the $\mathfrak{gl}(q|s)$ invariants $(k \circ l)$ are not sufficient to build up the non-compact Yangian invariant $|\Psi_{3,1}\rangle$. In fact,

[4] Let us mention a subtlety concerning the parameter K. It is defined as the number of negative energy representations in $\Psi_{2K,K}$. After selecting the representation labels $c_i = 0$, it agrees with the degree of helicity violation of the amplitude $\mathcal{A}_{2K,K}^{(\text{tree})}$, see section 1.3.3. In the context of amplitudes one usually does not specify the number of negative energy particles.

classical invariant theory, see e.g. [217], suggests to supplement these elementary building blocks by determinants of oscillators. Consequently, the oscillator dependence of the exponential function in the Graßmannian matrix model (4.24) will have to be modified for $N \neq 2K$. Another open question concerns formula (4.24) in the $N = 2K$ case. We pointed out the lack of efficient technology for its evaluation during the computation of sample invariants in section 4.1.5. We want to overcome this issue by applying matrix model methods for the evaluation of the Graßmannian integral (4.24) beyond the Leutwyler-Smilga case (4.22). One might wonder whether the Bessel function formula (4.23) generalizes to Yangian invariants with general deformation parameters v_i, c_i. One technique for this endeavor could be a character expansion, which was successfully employed for the Leutwyler-Smilga model in [180, 181]. Another auspicious method may be the use of Gelfand-Tzetlin coordinates, which has been applied to compute correlation functions of the Itzykson-Zuber model [218]. In our setting these coordinates might be well adapted to the minors appearing in the Graßmannian integral (4.1). Yet another approach could be to employ Weingarten functions [219], which provide explicit formulas for integrals of products of matrix elements over the unitary group. A different point to be addressed in the future is the transformation of the unitary Graßmannian matrix model (4.24) to twistor variables because these are used in a large part of the amplitudes literature. While we implemented the relation between oscillators and spinor helicity-like variables via a Bargmann transformation in this thesis, we completely left aside twistors. As the unitary contour is intrinsically related to Minkowski signature, we cannot simply employ the half Fourier transform of [67] for the transition from spinor helicity-like variables to twistors. However, a point of departure could be a twistorial description of the $\mathfrak{u}(p, q)$ oscillator representations, a.k.a. “ladder representations”, discussed e.g. in [220]. Let us move on to another promising topic. At the end of section 4.1.3 we mentioned that the partition functions of the Brezin-Gross-Witten and of the Leutwyler-Smilga model are solutions, so-called τ-functions, of the KP hierarchy, which is an infinite set of classically integrable equations. This immediately leads to the question whether also the more general unitary Graßmannian matrix model introduced in section 4.1.4 is a KP τ-function. Moreover, one might hope for a direct connection between the KP hierarchy and integrability in the sense of Yangian invariance because the latter was our guiding principle for the construction of the unitary matrix models in sections 4.1.3 and 4.1.4. Such a structural insight would make the large body of work on the KP hierarchy applicable for the construction of Yangian invariants and possibly also tree-level $\mathcal{N} = 4$ SYM scattering amplitudes. There is a further independent indication for a connection between Yangian invariants and the KP hierarchy. The partition function of the six-vertex model with domain wall boundary conditions is known to be a special KP τ-function [221]. In the rational limit this setup belongs to the class of vertex models considered in section 3.4. There we established that the partition functions of these models are components of Yangian invariant vectors. Additional clues might be provided by a link between transfer matrices of quantum integrable spin chains and τ-functions [222, 223]. Finally, we would like to discuss the relevance of our results for loop amplitudes. In section 1.3.6 we mentioned that a deformed amplitude $\mathcal{A}_{6,3}^{(\text{def.})}$ might yield a regularization of the one-loop amplitude $\mathcal{A}_{4,2}^{(1)}$. It remains to be seen if the deformed Yangian invariant $\Psi_{6,3}$ in (4.176) with (4.177) serves this purpose. Let us instead speculate about a conceptually clear route to loop-amplitudes. We already emphasized an important link between the unitary Graßmannian matrix model (4.24) for $\mathfrak{u}(2, 2|4)$ and the one-loop spectral problem of planar $\mathcal{N} = 4$ SYM. Both can be formulated in terms of essentially the same oscillator representations. We may use a grading for which each of the building blocks $(k \bullet l)$ and $(k \circ l)$ of (4.24) is invariant under one of the two compact subalgebras

$\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \subset \mathfrak{u}(2,2|2+2)$, see also appendix B.3.3. In the spectral problem a central extension of such subalgebras is the key to all-loop results, cf. section 1.2. Appealing to a common integrable structure of the entire $\mathcal{N}=4$ model, we suspect that in this way a coupling constant can also be introduced in our unitary Graßmannian integral formula. Such an approach would also shed light on an infinite-dimensional symmetry algebra that possibly governs the all-loop amplitudes.

After specifying our objectives in section 1.4, we set out in this dissertation to build a robust bridge between integrable models and planar $\mathcal{N}=4$ SYM scattering amplitudes. Even though this construction is most definitely not completed, we established promising ties between these two fields in both directions. The Bethe ansatz construction of simple Yangian invariants, which can be considered as toy models for amplitudes, is one example. The unitary Graßmannian integral with its origin in the field of amplitudes and its potential connection to integrable hierarchies is another one for the reverse direction. Our representation theoretic setup, that in particular respects the reality conditions of the variables involved, allows us to ask very detailed questions. Arguably the most urgent one at present is to clarify the relation of the Yangian invariants computed by our unitary Graßmannian integral and tree-level amplitudes. We hope to extend the framework established in this thesis in future work because we believe that it bears the potential to contribute to both areas of research, integrable models and scattering amplitudes.

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Appendix A

Loose Ends of Bethe Ansatz

In this appendix we take initial steps in addressing some questions that remained unresolved in the discussion of the Bethe ansatz for Yangian invariants in section 3.2. We begin in section A.1 by developing a technique to evaluate the Bethe vector leading to the compact two-site Yangian invariant for $\mathfrak{gl}(2)$. Section A.2 contains a generalization of the functional relations which characterize compact $\mathfrak{gl}(2)$ Yangian invariants to the $\mathfrak{gl}(n)$ case. Lastly, in section A.3 we demonstrate that for certain $\mathfrak{gl}(n)$ Yangian invariants the complicated nesting, which is inherent to higher rank Bethe ansätze, disappears.

A.1 Derivation of Two-Site Invariant

We constructed sample solutions of the functional relation (3.36) for finite-dimensional $\mathfrak{gl}(2)$ Yangian invariants in section 3.2.2. What is more, we even explained a classification of its solutions, in section 3.2.3. However, we did not present a satisfactory method to evaluate the corresponding Bethe vectors, although we proved them to be Yangian invariant. Instead, we relied on explicit case-by-case calculations for small values of the representation labels. Here we fill this gap for the simplest example of the two-site invariant $|\Psi_{2,1}\rangle$, which we discussed in section 3.2.2.1. The explicit form (2.61) of $|\Psi_{2,1}\rangle$ is a product of c_2 factors $(1 \bullet 2)$. Similarly, the associated algebraic Bethe vector (3.19) is a c_2-fold product of operators $B(u_k)$ with the Bethe roots u_k from (3.41). The idea of the following derivation is to keep this product structure manifest and to show how each factor $(1 \bullet 2)$ corresponds to one operator $B(u_k)$.

We reformulate the $\mathfrak{gl}(2)$ two-site Yangian invariant (2.61) as

$$
\begin{aligned}
|\Psi_{2,1}\rangle &= (1 \bullet 2)^{c_2}|0\rangle = (\bar{\mathbf{a}}_1^1\bar{\mathbf{a}}_1^2 + \bar{\mathbf{a}}_2^1\bar{\mathbf{a}}_2^2)^{c_2}|0\rangle \\
&= \left(\bar{\mathbf{a}}_1^1\mathbf{a}_2^1\frac{1}{\bar{\mathbf{a}}_2^1\mathbf{a}_2^1} + \bar{\mathbf{a}}_2^2\mathbf{a}_1^2\frac{1}{\bar{\mathbf{a}}_1^2\mathbf{a}_1^2}\right)^{c_2}(\bar{\mathbf{a}}_2^1\bar{\mathbf{a}}_1^2)^{c_2}|0\rangle = \left(\bar{\mathbf{J}}_{21}^1\frac{1}{\bar{\mathbf{J}}_{22}^1} + \mathbf{J}_{21}^2\frac{1}{\mathbf{J}_{11}^2}\right)^{c_2}|\Omega\rangle\,,
\end{aligned}
\tag{A.1}
$$

where the reference state $|\Omega\rangle$ is that of (3.42). The inverse powers of the number operators $\mathbf{J}_{11}^2$ and $\bar{\mathbf{J}}_{22}^1$ are well-defined because they act on states with at least one of the respective oscillators. Next, we move the inverse powers of $\mathbf{J}_{11}^2$ to the right using $\mathbf{J}_{11}^2\mathbf{J}_{21}^2 = \mathbf{J}_{21}^2(\mathbf{J}_{11}^2-1)$, which implies $f(\mathbf{J}_{11}^2)\mathbf{J}_{21}^2 = \mathbf{J}_{21}^2 f(\mathbf{J}_{11}^2-1)$ for a function $f(\mathbf{J}_{11}^2)$. Then we evaluate their action on the highest weight state with $\mathbf{J}^2_{11}|\Omega\rangle = c_2|\Omega\rangle$,

$$\begin{aligned}|\Psi_{2,1}\rangle = \frac{1}{c_2!}\left(\bar{\mathbf{J}}^1_{21}\frac{1}{\bar{\mathbf{J}}^1_{22}}\mathbf{J}^2_{11} + \mathbf{J}^2_{21}\right)\left(\bar{\mathbf{J}}^1_{21}\frac{1}{\bar{\mathbf{J}}^1_{22}}(\mathbf{J}^2_{11} - 1) + \mathbf{J}^2_{21}\right)\\ \cdots\left(\bar{\mathbf{J}}^1_{21}\frac{1}{\bar{\mathbf{J}}^1_{22}}(\mathbf{J}^2_{11} - c_2 + 1) + \mathbf{J}^2_{21}\right)|\Omega\rangle\,.\end{aligned} \tag{A.2}$$

As the factors in this product commute, we reverse their order. Next, the inverse powers of $\bar{\mathbf{J}}^1_{22}$ are moved to the right with the help of $\bar{\mathbf{J}}^1_{22}\bar{\mathbf{J}}^1_{21} = \bar{\mathbf{J}}^1_{21}(\bar{\mathbf{J}}^1_{22} + 1)$ and $\bar{\mathbf{J}}^1_{22}|\Omega\rangle = -c_2|\Omega\rangle$,

$$\begin{aligned}|\Psi_{2,1}\rangle = \frac{(-1)^{c_2}}{c_2!^2}\Big(\bar{\mathbf{J}}^1_{21}(\mathbf{J}^2_{11} - c_2 + 1) + \mathbf{J}^2_{21}\bar{\mathbf{J}}^1_{22}\Big)\\ \cdots\Big(\bar{\mathbf{J}}^1_{21}(\mathbf{J}^2_{11} - 1) + \mathbf{J}^2_{21}(\bar{\mathbf{J}}^1_{22} + c_2 - 2)\Big)\Big(\bar{\mathbf{J}}^1_{21}\mathbf{J}^2_{11} + J^2_{21}(\bar{\mathbf{J}}^1_{22} + c_2 - 1)\Big)|\Omega\rangle\,.\end{aligned} \tag{A.3}$$

Taking into account $\mathbf{J}^2_{11}+\mathbf{J}^2_{22} = c_2$ and $\bar{\mathbf{J}}^1_{11}+\bar{\mathbf{J}}^1_{22} = -c_2$, which is valid for the representations at hand, yields

$$\begin{aligned}|\Psi_{2,1}\rangle = \frac{1}{c_2!^2}\Big(\bar{\mathbf{J}}^1_{21}(\mathbf{J}^2_{22} - 1) + \mathbf{J}^2_{21}(\bar{\mathbf{J}}^1_{11} + c_2)\Big)\\ \cdots\Big(\bar{\mathbf{J}}^1_{21}(\mathbf{J}^2_{22} - c_2 + 1) + \mathbf{J}^2_{21}(\bar{\mathbf{J}}^1_{11} + 2)\Big)\Big(\bar{\mathbf{J}}^1_{21}(\mathbf{J}^2_{22} - c_2) + \mathbf{J}^2_{21}(\bar{\mathbf{J}}^1_{11} + 1)\Big)|\Omega\rangle\,.\end{aligned} \tag{A.4}$$

The Bethe vector (3.19) is built from the operator

$$B(u) = \frac{\bar{\mathbf{J}}^1_{21}(\mathbf{J}^2_{22} + u - v_2) + \mathbf{J}^2_{21}(\bar{\mathbf{J}}^1_{11} + u - v_1)}{(u - v_1)(u - v_2)}\,. \tag{A.5}$$

This is a matrix element of the monodromy (2.58) and we inserted the trivial normalization (2.60). Next, we use the two-site solution (3.38) of the functional relation (3.36), in particular $v_1 = v_2 - 1 - c_2$, and the Bethe roots (3.41), i.e. $u_k = v_2 - k$. This shows that each factor in (A.4) matches the numerator of one operator $B(u_k)$. The denominators account for the prefactor in (A.4). Consequently, we proved that

$$|\Psi_{2,1}\rangle = (1 \bullet 2)^{c_2}|0\rangle = (-1)^{c_2}B(u_1)\cdots B(u_{c_2-1})B(u_{c_2})|\Omega\rangle \tag{A.6}$$

for any $c_2 \in \mathbb{N}$. It would certainly be interesting to extend this derivation to the other sample invariants of section 3.2.2 and even more generally to the solutions of the functional relations classified in section 3.2.3.

A.2 Functional Equations for Higher Rank

In section 3.2.1 we showed that the algebraic Bethe ansatz for an inhomogeneous spin chain with finite-dimensional $\mathfrak{gl}(2)$ representations can be specialized to the case of Yangian invariant Bethe vectors. The Yangian invariants are then characterized by the functional relations (3.31). They restrict the admissible inhomogeneities and representations of the monodromy and furthermore determine the Bethe roots. Our method is based on the key observation that a Yangian invariant is a special eigenstate of a transfer matrix, cf. (3.26). This observation is clearly valid beyond the $\mathfrak{gl}(2)$ case, in particular it remains true for the higher rank $\mathfrak{gl}(n)$ algebra. Transfer matrices with finite-dimensional $\mathfrak{gl}(n)$ representations can be diagonalized using the *nested algebraic Bethe ansatz* in [160]. In the present section we specialize this construction to the case of Yangian invariant Bethe vectors. This leads to a $\mathfrak{gl}(n)$ generalization of the functional relations (3.31). We skip cumbersome technical details and do not present the derivation of these relations here, nor do we work out the intricate form of the associated Yangian invariant nested Bethe vectors.

The functional relations for $\mathfrak{gl}(n)$ read

$$\begin{aligned}
1 &= \mu_1(u)\frac{Q_1(u-1)}{Q_1(u)}\,,\\
1 &= \mu_2(u)\frac{Q_1(u+1)}{Q_1(u)}\frac{Q_2(u-1)}{Q_2(u)}\,,\\
1 &= \mu_3(u)\frac{Q_2(u+1)}{Q_2(u)}\frac{Q_3(u-1)}{Q_3(u)}\,,\\
&\vdots\\
1 &= \mu_{n-1}(u)\frac{Q_{n-2}(u+1)}{Q_{n-2}(u)}\frac{Q_{n-1}(u-1)}{Q_{n-1}(u)}\,,\\
1 &= \mu_n(u)\frac{Q_{n-1}(u+1)}{Q_{n-1}(u)}\,.
\end{aligned} \tag{A.7}$$

These relations restrict the inhomogeneities and the representation labels of $\mathfrak{gl}(n)$ monodromies $M(u)$ that admit Yangian invariants. Furthermore, they determine the corresponding Bethe roots. The monodromy matrix $M(u)$ is defined in (2.19) in terms of the Lax operators (2.18). The eigenvalues of its elements $M_{11}(u),\ldots,M_{nn}(u)$ on the reference state of the Bethe ansatz are denoted $\mu_1(u),\ldots,\mu_n(u)$, cf. (3.15) for the $\mathfrak{gl}(2)$ case. We work with a finite-dimensional $\mathfrak{gl}(n)$ representation $\mathcal{V}_i$ with a highest weight $\Xi_i=(\xi_i^{(1)},\ldots,\xi_i^{(n)})$ at the i-th spin chain site. In analogy to the $\mathfrak{gl}(2)$ equation (3.18), the monodromy eigenvalues are parametrized as

$$\mu_a(u)=\prod_{i=1}^{N} f_{\mathcal{V}_i}(u-v_i)\frac{u-v_i+\xi_i^{(a)}}{u-v_i}\,. \tag{A.8}$$

The Q-function

$$Q_k(u)=\prod_{i=1}^{P_k}(u-u_i^{(k)})\,, \tag{A.9}$$

encodes P_k Bethe roots $u_i^{(k)}$, where k is the nesting level taking the values $1,\ldots,n-1$. We observe that for $n=2$ the functional relations (A.7) reduce to the $\mathfrak{gl}(2)$ case (3.31).

The relations (A.7) decouple into

$$1=\prod_{a=1}^{n}\mu_a(u-a+1)\,, \tag{A.10}$$

$$\frac{Q_k(u)}{Q_k(u+1)}=\prod_{a=k+1}^{n}\mu_a(u-a+k+1) \tag{A.11}$$

with $k=1,\ldots,n-1$. Here (A.10) is free of Bethe roots and solely constraints the inhomogeneities and representations of the monodromy. Equation (A.11) contains only Bethe roots of the nesting level k. The equations (A.10) and (A.11) generalize the respective $\mathfrak{gl}(2)$ versions (3.36) and (3.37). We may think of (A.10) as the most important result of this section because its constraints on $\mathfrak{gl}(n)$ monodromies admitting Yangian invariants are valid independent of the Bethe ansatz construction.

A.3 Higher Rank Invariants from Bethe Vectors

The algebraic Bethe ansatz for $\mathfrak{gl}(2)$ spin chains uses a monodromy $M(u)$ that is a 2×2 matrix in the auxiliary space, see section 3.1. The Bethe vectors $|\Psi\rangle$ are products of the monodromy element $M_{12}(u) \equiv B(u)$ acting on a reference state $|\Omega\rangle$. Consequently, also the Yangian invariant Bethe vectors in section 3.2 are of this form. In case of the algebra $\mathfrak{gl}(n)$ the monodromy matrix $M(u)$ in (2.19) is a $n \times n$ matrix in the auxiliary space. The nested Bethe ansatz construction [160] of a general Bethe vector $|\Psi\rangle$ involves contributions from all monodromy elements $M_{\alpha\beta}(u)$ with $\alpha < \beta$. This leads to rather involved formulas for these vectors, which is why we did not include them in section A.2. Here we show, by way of example, that certain $\mathfrak{gl}(n)$ Yangian invariants $|\Psi\rangle$ can be expressed as the action of only operators $M_{1n}(u)$ on a reference state $|\Omega\rangle$. Thereby the usual nesting procedure is bypassed completely.

Once again, we employ the compact $\mathfrak{gl}(n)$ oscillators representations $\mathcal{D}_c$ and $\bar{\mathcal{D}}_{-c}$ with $c \in \mathbb{N}$ of section 2.4.1.1 at the sites of the monodromy $M(u)$. In contrast to the sample invariants considered in section 2.4.1, we place the sites with dual representations $\bar{\mathcal{D}}_{-c}$ right of those with ordinary representations $\mathcal{D}_c$. This order seems to be necessary for the simple structure of the Yangian invariants $|\Psi\rangle$ that we are aiming at. In addition, to find solutions of the Yangian invariance condition (2.24) for such monodromies, we work with the non-trivial normalization

$$f_{\mathcal{V}_i}(u - v_i) = \frac{u - v_i}{u - v_i - 1} \tag{A.12}$$

of the Lax operators (2.18).

We remark that the simple expressions for the sample invariants, which we will present shortly, are at present purely based on explicit calculations. It would be desirable to understand how they come about as a reduction of the nested Bethe vectors of [160] and thereby also establish a connection with section A.2. Alternatively, one could attempt to show the Yangian invariance of our expressions directly using the commutation relations (2.9) of the monodromy elements. Either approach should lead to a better understanding of the hidden simplicity of Yangian invariant Bethe vectors. Let us now present our list of sample invariants.

A.3.1 Two-Site Invariant and Identity Operator

We introduce the $\mathfrak{gl}(n)$ monodromy matrix

$$M(u) = R_{\square\,\mathcal{D}_{c_1}}(u - v_1) R_{\square\,\bar{\mathcal{D}}_{c_2}}(u - v_2) \tag{A.13}$$

with

$$v_1 = v_2 + c_1 - 1\,, \quad c_1 + c_2 = 0\,. \tag{A.14}$$

A computation along the lines of that for the $\mathfrak{gl}(2)$ invariant in section A.1 shows

$$|\Psi\rangle = M_{1n}(u_1) \cdots M_{1n}(u_P)|\Omega\rangle \propto (1 \bullet 2)^{c_1}|0\rangle\,, \tag{A.15}$$

where $|\Omega\rangle = (\bar{\mathbf{a}}_1^1)^{c_1}(\bar{\mathbf{a}}_n^2)^{c_1}|0\rangle$ and the $P = c_1$ "Bethe roots" are given by

$$u_k = v_2 + k - 1\,. \tag{A.16}$$

We use quotation marks to remind the reader that we did not derive the form of the vector (A.15) from a Bethe ansatz. Anyhow, (A.15) satisfies the Yangian invariance condition (2.24), as one easily checks by a direct computation. Notice that (A.15) is an element of $\mathcal{D}_{c_1} \otimes \bar{\mathcal{D}}_{c_2}$ and therefore differs from $|\Psi_{2,1}\rangle$ of section 2.4.1.3 which is in $\bar{\mathcal{D}}_{c_1} \otimes \mathcal{D}_{c_2}$. Nevertheless, we may interpret also the intertwiner version of (A.15) as an identity operator.

A.3.2 Four-Site Invariant and R-Matrix

We consider the monodromy matrix

$$M(u) = R_{\square\,\mathcal{D}_{c_1}}(u-v_1)R_{\square\,\mathcal{D}_{c_2}}(u-v_2)R_{\square\,\bar{\mathcal{D}}_{c_3}}(u-v_3)R_{\square\,\bar{\mathcal{D}}_{c_4}}(u-v_4) \tag{A.17}$$

with

$$v_2 = v_4 + c_2 - 1\,, \quad c_2 + c_4 = 0\,, \quad v_1 = v_3 + c_1 - 1\,, \quad c_1 + c_3 = 0\,. \tag{A.18}$$

A case-by-case computation for small values of the representation labels c_1, c_2 yields

$$\begin{aligned} |\Psi\rangle &= M_{1n}(u_1)\cdots M_{1n}(u_P)|\Omega\rangle \\ &\propto \sum_{k=0}^{\min(c_1,c_2)} \frac{k!}{\Gamma(v_3-v_4-c_2+k+1)} \frac{(1\bullet 3)^{c_1-k}}{(c_1-k)!} \frac{(2\bullet 4)^{c_2-k}}{(c_2-k)!} \frac{(1\bullet 4)^{k}}{k!} \frac{(2\bullet 3)^{k}}{k!} |0\rangle\,, \end{aligned} \tag{A.19}$$

where $|\Omega\rangle = (\bar{\mathbf{a}}_1^1)^{c_1}(\bar{\mathbf{a}}_1^2)^{c_2}(\bar{\mathbf{a}}_n^3)^{c_1}(\bar{\mathbf{a}}_n^4)^{c_2}|0\rangle$ and we have $P = c_1 + c_2$ "Bethe roots" given by

$$\begin{aligned} u_k &= v_3 + k - 1 \quad \text{for} \quad k = 1, \ldots, c_1\,, \\ u_{k+c_1} &= v_4 + k - 1 \quad \text{for} \quad k = 1, \ldots, c_2\,. \end{aligned} \tag{A.20}$$

By means of a direct calculation we verify that (A.19) solves the Yangian invariance condition (2.24). The intertwiner version of this condition is a Yang-Baxter equation. Thus the invariant (A.19) corresponds to a $\mathfrak{gl}(n)$ R-matrix. Furthermore, it is akin to the invariant $|\Psi_{4,2}(z)\rangle$ investigated in section 2.4.1.5.

A.3.3 Another Four-Site Invariant and Identity Operators

We use a monodromy of the same form as in (A.17),

$$M(u) = R_{\square\,\mathcal{D}_{c_1}}(u-v_1)R_{\square\,\mathcal{D}_{c_2}}(u-v_2)R_{\square\,\bar{\mathcal{D}}_{c_3}}(u-v_3)R_{\square\,\bar{\mathcal{D}}_{c_4}}(u-v_4)\,. \tag{A.21}$$

This time, however, the parameters obey the constraints

$$v_1 = v_4 + c_1 - 1\,, \quad c_1 + c_4 = 0\,, \quad v_2 = v_3 + c_2 - 1\,, \quad c_2 + c_3 = 0\,. \tag{A.22}$$

For small values of the representation labels we compute

$$|\Psi\rangle = M_{1n}(u_1)\cdots M_{1n}(u_P)|\Omega\rangle \propto (1\bullet 4)^{c_1}(2\bullet 3)^{c_2}|0\rangle\,, \tag{A.23}$$

where $|\Omega\rangle = (\bar{\mathbf{a}}_1^1)^{c_1}(\bar{\mathbf{a}}_1^2)^{c_2}(\bar{\mathbf{a}}_n^3)^{c_2}(\bar{\mathbf{a}}_n^4)^{c_1}|0\rangle$ and there are $P = c_1 + c_2$ "Bethe roots"

$$\begin{aligned} u_k &= v_4 + k - 1 \quad \text{for} \quad k = 1, \ldots, c_1\,, \\ u_{k+c_1} &= v_3 + k - 1 \quad \text{for} \quad k = 1, \ldots, c_2\,. \end{aligned} \tag{A.24}$$

The Yangian invariance condition (2.24) for (A.23) is checked by a direct calculation. Comparing with the two-site invariant in section A.3.1, we can interpret the intertwiner version of (A.23) as the product of two identity operators.

A.3.4 Yet Another Four-Site Invariant

Let us choose the monodromy

$$M(u) = R_{\square\,\mathcal{D}_{c_1}}(u - v_1)R_{\square\,\bar{\mathcal{D}}_{c_2}}(u - v_2)R_{\square\,\mathcal{D}_{c_3}}(u - v_3)R_{\square\,\bar{\mathcal{D}}_{c_4}}(u - v_4) \tag{A.25}$$

with

$$v_1 = v_2 + c_1 - 1\,, \quad c_1 + c_2 = 0\,, \quad v_3 = v_4 + c_3 - 1\,, \quad c_3 + c_4 = 0\,. \tag{A.26}$$

Again we do a computation for small representation labels to obtain

$$|\Psi\rangle = M_{1n}(u_1)\cdots M_{1n}(u_P)|\Omega\rangle \propto (1 \bullet 2)^{c_1}(3 \bullet 4)^{c_3}|0\rangle\,, \tag{A.27}$$

where $|\Omega\rangle = (\bar{\mathbf{a}}_1^1)^{c_1}(\bar{\mathbf{a}}_n^2)^{c_1}(\bar{\mathbf{a}}_1^3)^{c_3}(\bar{\mathbf{a}}_n^4)^{c_3}|0\rangle$ and we have the $P = c_1 + c_3$ "Bethe roots"

$$\begin{aligned} u_k &= v_2 + k - 1 \quad \text{for} \quad k = 1, \ldots, c_1\,, \\ u_{k+c_1} &= v_4 + k - 1 \quad \text{for} \quad k = 1, \ldots, c_3\,. \end{aligned} \tag{A.28}$$

Also here we fall back to an explicit calculation to show the Yangian invariance (2.24) of the vector (A.27). We included this example because in the monodromy (A.25) not all dual sites are right of the ordinary ones. However, this constraint on the order of sites still holds within those that are linked by (A.26), i.e. $\bar{\mathcal{D}}_{c_2}$ is right of $\mathcal{D}_{c_1}$ and $\bar{\mathcal{D}}_{c_4}$ is right of $\mathcal{D}_{c_3}$.

Appendix B

Additional Material on Graßmannian Integral

This appendix contains supplementary results on the unitary Graßmannian integral introduced in chapter 4. In section B.1 we show that the $U(3)$ contour of the bosonic oscillator Yangian invariant $|\Psi_{6,3}\rangle$ emerges from "gluing" the $U(2)$ contours of three invariants of the type $|\Psi_{4,2}\rangle$. Section B.2 deals with a discrete parity symmetry of the unitary Graßmannian integral. It interrelates some of the kinematic regions which we encountered for the sample invariants in terms of spinor helicity-like variables in section 4.3.3. Lastly, we present some additional instructive sample invariants in terms of these variables in section B.3.

B.1 Gluing of Contours

The invariant $|\Psi_{6,3}\rangle$ can be obtained by combining, let's say "gluing together", three invariants of the type $|\Psi_{4,2}\rangle$. Another way of phrasing this is that the intertwiner version of $|\Psi_{6,3}\rangle$ is the product of three R-matrices, cf. sections 2.4.1.2 and 2.4.1.5. In the vertex model language of section 3.4 such an invariant is associated with a Baxter lattice consisting of three lines with three intersections, each of which represents one R-matrix. We discussed intertwiners as well as Baxter lattices in detail only for compact bosonic algebras. Yet the assertion in the first sentence of this paragraph is valid more generally. Here we construct the invariant $|\Psi_{6,3}\rangle$ from three invariants of the type $|\Psi_{4,2}\rangle$ for the non-compact algebra $\mathfrak{u}(p,q|0)$ using the framework of chapter 4. We restrict to the bosonic case for clarity. Importantly, we show that this construction is compatible with the unitary contour of the Graßmannian matrix model (4.24) with the integrand (4.30). We demonstrate that the three $U(2)$ contours of the invariants $|\Psi_{4,2}\rangle$ combine into one $U(3)$ contour of $|\Psi_{6,3}\rangle$. The procedure described here should be thought of as analogue of the gluing of on-shell diagrams for $\mathcal{N}=4$ SYM scattering amplitudes, cf. sections 1.3.5 and 1.3.6. In contrast to our approach, the gluing of the contour is usually neglected for amplitudes.

Let us show that the $U(3)$ integral (4.24) with the integrand (4.33) for the invariant $|\Psi_{6,3}\rangle$ can be obtained from the vector

$$\Big(|\Psi^{(3)}_{4,2}\rangle\Big)^{\dagger_8\dagger_9}\Big(|\Psi^{(2)}_{4,2}\rangle\Big)^{\dagger_7}|\Psi^{(1)}_{4,2}\rangle\,. \tag{B.1}$$

Here the three invariants of type $|\Psi_{4,2}\rangle$ are given by the integral (4.24) with the $U(2)$ matrices

$$\mathcal{C}^{(1)}=\begin{pmatrix}C_{17} & C_{18}\\ C_{27} & C_{28}\end{pmatrix}\,,\quad \mathcal{C}^{(2)}=\begin{pmatrix}C_{74} & C_{79}\\ C_{34} & C_{39}\end{pmatrix}\,,\quad \mathcal{C}^{(3)}=\begin{pmatrix}C_{85} & C_{86}\\ C_{95} & C_{96}\end{pmatrix}\,, \tag{B.2}$$

the oscillator contractions

$$\mathbf{I}^{(1)}_{\overset{\bullet}{\circ}} = \begin{pmatrix} (1 \overset{\bullet}{\circ} 7) & (1 \overset{\bullet}{\circ} 8) \\ (2 \overset{\bullet}{\circ} 7) & (2 \overset{\bullet}{\circ} 8) \end{pmatrix} , \quad \mathbf{I}^{(2)}_{\overset{\bullet}{\circ}} = \begin{pmatrix} (7 \overset{\bullet}{\circ} 4) & (7 \overset{\bullet}{\circ} 9) \\ (3 \overset{\bullet}{\circ} 4) & (3 \overset{\bullet}{\circ} 9) \end{pmatrix} , \quad \mathbf{I}^{(3)}_{\overset{\bullet}{\circ}} = \begin{pmatrix} (8 \overset{\bullet}{\circ} 5) & (8 \overset{\bullet}{\circ} 6) \\ (9 \overset{\bullet}{\circ} 5) & (9 \overset{\bullet}{\circ} 6) \end{pmatrix} \tag{B.3}$$

and the integrands (4.32)

$$\begin{aligned} \mathscr{F}^{(1)}(\mathcal{C}^{(1)})^{-1} &= (-1)^{c_1+c_2}(\det \mathcal{C}^{(1)})^{q-c_2}|[1]_{\mathcal{C}^{(1)}}|^{2(1+v_1-v_2)}([1]_{\mathcal{C}^{(1)}})^{c_2-c_1} , \\ \mathscr{F}^{(2)}(\mathcal{C}^{(2)})^{-1} &= (-1)^{c_4+c_3}(\det \mathcal{C}^{(2)})^{q-c_3}|[1]_{\mathcal{C}^{(2)}}|^{2(1+v_4-n+1-c_4-v_3)}([1]_{\mathcal{C}^{(2)}})^{c_3+c_4} , \\ \mathscr{F}^{(3)}(\mathcal{C}^{(3)})^{-1} &= (-1)^{c_5+c_6}(\det \mathcal{C}^{(3)})^{q+c_6}|[1]_{\mathcal{C}^{(3)}}|^{2(1+v_5-v_6-c_5+c_6)}([1]_{\mathcal{C}^{(3)}})^{c_5-c_6} . \end{aligned} \tag{B.4}$$

The expression (B.1) contains oscillators at sites $7, 8, 9$ only in inner products. Thus it is a vector in the tensor product of the spaces associated with the sites $1, \ldots, 6$. In (B.4) we used (4.29) for $|\Psi_{4,2}\rangle$ to express the integrands in terms of the parameters v_i and c_i of only those spaces. We can write (B.1) as

$$\begin{aligned} &\left(|\Psi^{(3)}_{4,2}\rangle\right)^{\dagger_8\dagger_9}\left(|\Psi^{(2)}_{4,2}\rangle\right)^{\dagger_7}|\Psi^{(1)}_{4,2}\rangle \\ &\quad = \chi_2^{-3}\int_{U(2)}[\mathrm{d}\mathcal{C}^{(1)}]\int_{U(2)}[\mathrm{d}\mathcal{C}^{(2)}]\int_{U(2)}[\mathrm{d}\mathcal{C}^{(3)}]|[1]_{\mathcal{C}^{(2)}}|^2\mathscr{F}(\mathcal{C})e^{\operatorname{tr}(\mathcal{C}\mathbf{I}^t_{\bullet}+\mathbf{I}_{\circ}\mathcal{C}^\dagger)}|0\rangle , \end{aligned} \tag{B.5}$$

where $\mathscr{F}(\mathcal{C})$ is already the integrand of $|\Psi_{6,3}\rangle$ given in (4.33). Furthermore,

$$\mathcal{C} = \begin{pmatrix} C_{17} & C_{18} & 0 \\ C_{27} & C_{28} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_{74} & 0 & C_{79} \\ 0 & 1 & 0 \\ C_{34} & 0 & C_{39} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{85} & C_{86} \\ 0 & C_{95} & C_{96} \end{pmatrix} \tag{B.6}$$

and

$$\mathbf{I}_{\overset{\bullet}{\circ}} = \begin{pmatrix} (1 \overset{\bullet}{\circ} 4) & (1 \overset{\bullet}{\circ} 5) & (1 \overset{\bullet}{\circ} 6) \\ (2 \overset{\bullet}{\circ} 4) & (2 \overset{\bullet}{\circ} 5) & (2 \overset{\bullet}{\circ} 6) \\ (3 \overset{\bullet}{\circ} 4) & (3 \overset{\bullet}{\circ} 5) & (3 \overset{\bullet}{\circ} 6) \end{pmatrix} . \tag{B.7}$$

Here we eliminated the oscillators $\mathbf{A}^i_{\mathcal{A}}$ and $\bar{\mathbf{A}}^i_{\mathcal{A}}$ at sites $i = 7, 8, 9$ from (B.1) using $\langle 0|e^{\mathbf{B}\mathbf{A}^i_{\mathcal{A}}}e^{\mathbf{C}\bar{\mathbf{A}}^i_{\mathcal{A}}}|0\rangle = e^{\mathbf{BC}}$ for commuting operators $\mathbf{B}$ and $\mathbf{C}$. In addition, we used (4.27) to relate the minors of the $U(2)$ matrices $\mathcal{C}^{(1)}$, $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(3)}$ to those of the $U(3)$ matrix $\mathcal{C}$. We also made use of the relation (4.29) for the parameters v_i and c_i of $|\Psi_{6,3}\rangle$. Finally, we can identify (B.5) with the $U(3)$ integral for $|\Psi_{6.3}\rangle$ in (4.24),

$$i(2\pi)^{-3}\left(|\Psi^{(3)}_{4,2}\rangle\right)^{\dagger_8\dagger_9}\left(|\Psi^{(2)}_{4,2}\rangle\right)^{\dagger_7}|\Psi^{(1)}_{4,2}\rangle = \chi_3^{-1}\int_{U(3)}[\mathrm{d}\mathcal{C}]\mathscr{F}(\mathcal{C})e^{\operatorname{tr}(\mathcal{C}\mathbf{I}^t_{\bullet}+\mathbf{I}_{\circ}\mathcal{C}^\dagger)}|0\rangle = |\Psi_{6,3}\rangle . \tag{B.8}$$

For this step we parameterized each of the $U(2)$ matrices $\mathcal{C}^{(1)}$, $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(3)}$ as in (4.39) with variables $\theta^{(1)}, \alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}$ etc. Then we introduced the new variables

$$\begin{gathered} \alpha_1 = \alpha^{(1)} + \alpha^{(2)} - \gamma^{(3)} , \quad \alpha_3 = \alpha^{(3)} + \alpha^{(2)} , \quad \gamma = \gamma^{(1)} + \gamma^{(2)} + \gamma^{(3)} , \\ \beta_1 = \beta^{(1)} - \alpha^{(2)} + \gamma^{(3)} , \quad \beta_2 = \beta^{(2)} + \gamma^{(1)} + \gamma^{(3)} , \quad \beta_3 = \beta^{(3)} + \alpha^{(2)} - \gamma^{(3)} , \\ \theta_1 = \theta^{(1)} , \quad \theta_2 = \theta^{(2)} , \quad \theta_3 = \theta^{(3)} , \end{gathered} \tag{B.9}$$

which bring the $U(3)$ matrix $\mathcal{C}$ of (B.6) into the form given in (4.42). We reparameterized the integral (B.5) in terms of these new variables together with $\alpha^{(2)}$, $\gamma^{(2)}$ and $\gamma^{(3)}$. The integrals in the latter three variables only contribute a factor of 2π each because these variables do not appear in the integrand of (B.5) anymore. The remaining integrals of the three $U(2)$ Haar measures and the one minor of $\mathcal{C}^{(2)}$ in (B.5) combine into the $U(3)$ Haar measure (4.46). This concludes our calculation.

We add some comments. The parameterization of $U(3)$ by $U(2)$ matrices in (4.42) is well adapted to the gluing procedure because we just saw that each invariant of the type $|\Psi_{4,2}\rangle$ is associated with one of the $U(2)$ matrices. Furthermore, we can argue that once the contour of the Graßmannian integral (4.1) is fixed to be $U(2)$ for the invariant $|\Psi_{4,2}\rangle$, the gluing implies that we have to choose $U(3)$ for $|\Psi_{6,3}\rangle$. Moreover, it should be possible to construct the general invariant $|\Psi_{2K,K}\rangle$ from gluing. In that case we should end up with a parameterization of $U(K)$ in terms of $U(2)$ matrices as in [191]. We expect our calculation to generalize straightforwardly to superalgebras $\mathfrak{u}(p,q|r+s)$, except for the usual tedious subtleties with signs.

B.2 Discrete Parity Symmetry

In this section we investigate a discrete symmetry transformation of the unitary Graßmannian integral formulas (4.24) in oscillator variables and (4.111) in terms of spinor helicity-like variables. What is more, we study this symmetry explicitly for some sample invariants. The investigation of the oscillator invariant $|\Psi_{4,2}\rangle$ shows that for this example it is identical to the "parity" symmetry of the corresponding R-matrix. Interestingly, when applied to the invariants $\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ for $\mathfrak{u}(1,1)$ and $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ for $\mathfrak{u}(2,2)$, this symmetry transformation relates some of the kinematic regions that we encountered for these invariants in section 4.3.3.

B.2.1 On Level of Graßmannian Integral

We define the parity transformation $\mathcal{P}$ of the Graßmannian integral (4.24) for $\mathfrak{u}(p,q|r+s)$ oscillator Yangian invariants by reversing the order of the oscillators at the dual sites as well as at the ordinary sites,

$$\bar{\mathbf{A}}^i_{\mathcal{A}} \overset{\mathcal{P}}{\mapsto} \begin{cases} \bar{\mathbf{A}}^{K-i+1}_{\mathcal{A}} & \text{for} \quad i=1,\ldots,K\,, \\ \bar{\mathbf{A}}^{2K+K-i+1}_{\mathcal{A}} & \text{for} \quad i=K+1,\ldots,2K\,. \end{cases} \tag{B.10}$$

For the matrices of oscillator contractions entering (4.24) this translates into

$$\mathbf{I}_{\substack{\bullet\\\circ}} \overset{\mathcal{P}}{\mapsto} \mathscr{E}\,\mathbf{I}_{\substack{\bullet\\\circ}}\,\mathscr{E} \quad \text{with} \quad \mathscr{E}=\mathscr{E}^{-1}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in U(K)\,. \tag{B.11}$$

Using the left- and right-invariance of the Haar measure we find that the Graßmannian integral (4.24) is invariant under the transformation (B.10),

$$|\Psi_{2K,K}\rangle \overset{\mathcal{P}}{\mapsto} \chi_K^{-1}\int_{U(K)}[\mathrm{d}\mathcal{C}]\,\mathscr{F}(\mathscr{E}\mathcal{C}\mathscr{E})\,e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^\dagger)}|0\rangle = |\Psi_{2K,K}\rangle\,, \tag{B.12}$$

provided that $\mathscr{F}(\mathscr{E}\mathcal{C}\mathscr{E})=\mathscr{F}(\mathcal{C})$. This constraint is satisfied if the parameters in the function $\mathscr{F}(\mathcal{C})$ defined in (4.30) obey

$$\begin{aligned} v_1-v_2 &= v_{K-1}-v_K+c_K-c_{K-1}\,, &\quad \ldots\,, &\quad v_{K-1}-v_K = v_1-v_2+c_2-c_1\,, \\ c_2 &= c_{K-1}\,, & \ldots\,, &\quad c_K = c_1\,. \end{aligned} \tag{B.13}$$

Here we used $[1,2,\ldots,j]_{\mathscr{E}\mathcal{C}\mathscr{E}} = \overline{[1,2,\ldots,K-j]}_{\mathcal{C}} \det \mathcal{C}$ for the minors appearing in $\mathscr{F}(\mathscr{E}\mathcal{C}\mathscr{E})$. Notice that a particular solution of the constraints in (B.13) is given by

$$v_1 - v_2 = v_2 - v_3 = \cdots = v_{K-1} - v_K\,, \quad c_1 = c_2 = \cdots = c_K\,. \tag{B.14}$$

This discussion of the parity symmetry easily translates to the Graßmannian integral (4.111) for $\mathfrak{u}(p,p)$ in spinor helicity-like variables, where we restrict to bosonic algebras to avoid nasty sign factors. With the change of variables from oscillators to these variables given in (4.53), (4.68) and (4.70), the transformation (B.10) reads

$$\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}^{\mathrm{d}} \\ \hdashline \boldsymbol{\lambda}^{\mathrm{o}} \end{pmatrix} \overset{\mathcal{P}}{\mapsto} \begin{pmatrix} \mathscr{E}\boldsymbol{\lambda}^{\mathrm{d}} \\ \hdashline \mathscr{E}\boldsymbol{\lambda}^{\mathrm{o}} \end{pmatrix}. \tag{B.15}$$

Similar to the oscillator case one verifies that the Graßmannian integral (4.111) is invariant under this transformation,

$$\Psi_{2K,K}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}}) \overset{\mathcal{P}}{\mapsto} \Psi_{2K,K}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})\,, \tag{B.16}$$

if the parameters obey (B.13).

B.2.2 On Level of Sample Invariants

B.2.2.1 Four-Site Invariant in Oscillator Variables

Let us study the parity transformation $\mathcal{P}$ more explicitly for the oscillator invariant $|\Psi_{4,2}\rangle$ defined by the Graßmannian integral (4.24). In this case the transformation (B.11) becomes

$$\mathbf{I}_{\overset{\bullet}{\circ}} = \begin{pmatrix} (1 \overset{\bullet}{\circ} 3) & (1 \overset{\bullet}{\circ} 4) \\ (2 \overset{\bullet}{\circ} 3) & (2 \overset{\bullet}{\circ} 4) \end{pmatrix} \overset{\mathcal{P}}{\mapsto} \begin{pmatrix} (2 \overset{\bullet}{\circ} 4) & (2 \overset{\bullet}{\circ} 3) \\ (1 \overset{\bullet}{\circ} 4) & (1 \overset{\bullet}{\circ} 3) \end{pmatrix}. \tag{B.17}$$

According to (B.14) the Yangian invariant $|\Psi_{4,2}\rangle$ is parity invariant for $c_1 = c_2 \in \mathbb{Z}$ and unconstrained $v_1, v_2 \in \mathbb{C}$. Of course, this property can also be verified on the level of the explicit expression (4.49) for the $\mathfrak{u}(p,q|r+s)$ version of $|\Psi_{4,2}\rangle$. We do not detail this computation here and just mention that it is important to pay close attention to the constraints of the summation range in (4.50). However, it is helpful to take a look at the compact $\mathfrak{u}(n)$ special case of (4.49) that we discussed in detail in section 2.4.1.5, albeit with a different normalization. In this case $|\Psi_{4,2}\rangle$ is given in (2.79) as a sum over terms specified in (2.80). We immediately see that each of these terms is invariant under the transformation (B.17). In the aforementioned section we also reformulated $|\Psi_{4,2}\rangle$ as an R-matrix acting on the tensor product $\mathcal{D}_{-c_1} \otimes \mathcal{D}_{-c_2}$. For this R-matrix the invariance under the transformation (B.17) translates into the invariance under the permutation of tensor factors. In the literature on integrable models this property is known as "parity invariance", see e.g. [224]. Therefore we also refer to the general transformation $\mathcal{P}$ defined in section B.2.1 as parity transformation.

B.2.2.2 Four-Site Invariant for $\mathfrak{u}(1,1)$ in Spinor Helicity Variables

We move on to discuss the Yangian invariant $\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ for $\mathfrak{u}(1,1)$, which we constructed in section 4.3.3.3 using the unitary Graßmannian integral (4.111). According to section B.2.1 it is invariant under the parity transformation (B.15) if the representation labels satisfy $c_1 = c_2$. Let us in addition restrict the deformation parameters to $v_1 = v_2$. In this case $\Psi_{4,2}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ is given by (4.131) with (4.142). The latter equation shows that for a fixed value of $c_1 = c_2$ there are two kinematic regions in which the expression for the invariant differs. These regions are characterized by the absolute values of the variables $\mathtt{A}$ and $\mathtt{B}$ defined in (4.135). It is instructive to study the action of the parity transformation (B.15) on these regions. For the invariant under consideration this transformation reads

$$\begin{pmatrix} \lambda_1^1 \\ \lambda_1^2 \end{pmatrix} \overset{\mathcal{P}}{\mapsto} \begin{pmatrix} \lambda_1^2 \\ \lambda_1^1 \end{pmatrix} , \quad \begin{pmatrix} \lambda_1^3 \\ \lambda_1^4 \end{pmatrix} \overset{\mathcal{P}}{\mapsto} \begin{pmatrix} \lambda_1^4 \\ \lambda_1^3 \end{pmatrix} . \tag{B.18}$$

It implies

$$\mathtt{A} \overset{\mathcal{P}}{\mapsto} -\overline{\mathtt{B}} \,, \quad \mathtt{B} \overset{\mathcal{P}}{\mapsto} -\overline{\mathtt{A}} \,. \tag{B.19}$$

Consequently, the two kinematic regions and, respectively, the associated expressions for the invariant in (4.142) are interchanged,

$$|\mathtt{A}| > |\mathtt{B}| \quad \overset{\mathcal{P}}{\leftrightarrow} \quad |\mathtt{A}| < |\mathtt{B}| \,. \tag{B.20}$$

Thus the existence of these two regions in (4.142) is of crucial importance for the parity symmetry of $\Psi_{4,2}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ for $\mathfrak{u}(1,1)$.

Let us discuss what happens if we use the expression of either region in (4.142) and declare it to be valid for all values of $\mathtt{A}$ and $\mathtt{B}$. The resulting function still satisfies the Yangian invariance condition (2.24), which takes the form of a differential equation in λ_1^i, for generic values of λ_1^i. However, it violates parity symmetry. Hence this function cannot be the correct evaluation of the unitary Graßmannian integral (4.111). Moreover, it cannot be related to the oscillator Yangian invariant $|\Psi_{4,2}\rangle$ for $\mathfrak{u}(1,1)$ via the change of basis introduced in section 4.2 because $|\Psi_{4,2}\rangle$ is parity symmetric.

B.2.2.3 Six-Site Invariant for $\mathfrak{u}(2,2)$ in Spinor Helicity Variables

In section 4.3.3.5 we constructed the Yangian invariant $\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ for $\mathfrak{u}(2,2)$ from the unitary Graßmannian integral (4.111). Here we focus on the case with representation labels $c_1 = c_2 = c_3$ and deformation parameters $v_1 = v_2 = v_3$. Section B.2.1 shows that in this case the Yangian invariant $\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ is also invariant under the parity transformation (B.15). $\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ is given explicitly by (4.159) with (4.170) as a sum of residues. The selection of residues differs for four kinematic regions, which are characterized by the absolute values of the variables $\mathtt{A}, \mathtt{D}, \mathtt{C}, \mathtt{D}$ defined in (4.165). After (4.170) we explained that this characterization can be translated into conditions on the generalized Mandelstam variables s_{126} and s_{234}. We want to study the action of the parity transformation (B.15),

$$\begin{pmatrix} \lambda_\alpha^1 \\ \lambda_\alpha^2 \\ \lambda_\alpha^3 \end{pmatrix} \overset{\mathcal{P}}{\mapsto} \begin{pmatrix} \lambda_\alpha^3 \\ \lambda_\alpha^2 \\ \lambda_\alpha^1 \end{pmatrix} , \quad \begin{pmatrix} \lambda_\alpha^4 \\ \lambda_\alpha^5 \\ \lambda_\alpha^6 \end{pmatrix} \overset{\mathcal{P}}{\mapsto} \begin{pmatrix} \lambda_\alpha^6 \\ \lambda_\alpha^5 \\ \lambda_\alpha^4 \end{pmatrix} , \tag{B.21}$$

on this explicit form of $\Psi_{6,3}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$. First, we observe

$$\mathtt{A} \overset{\mathcal{P}}{\mapsto} \overline{\mathtt{C}} \,, \quad \mathtt{B} \overset{\mathcal{P}}{\mapsto} \overline{\mathtt{D}} \,, \quad \mathtt{C} \overset{\mathcal{P}}{\mapsto} \overline{\mathtt{A}} \,, \quad \mathtt{D} \overset{\mathcal{P}}{\mapsto} \overline{\mathtt{B}} \,. \tag{B.22}$$

From this we compute the transformation of the individual residues appearing in (4.170),

$$\begin{aligned} &\mathrm{res}_0 \mathscr{I}(u) \overset{\mathcal{P}}{\mapsto} \mathrm{res}_0 \mathscr{I}(u) \,, \quad \mathrm{res}_\infty \mathscr{I}(u) \overset{\mathcal{P}}{\mapsto} \mathrm{res}_\infty \mathscr{I}(u) \,, \\ &\mathrm{res}_{\frac{\mathtt{A}}{\mathtt{B}}} \mathscr{I}(u) \overset{\mathcal{P}}{\leftrightarrow} \mathrm{res}_{\frac{\overline{\mathtt{C}}}{\overline{\mathtt{D}}}} \mathscr{I}(u) \,, \quad \mathrm{res}_{\frac{\overline{\mathtt{B}}}{\overline{\mathtt{A}}}} \mathscr{I}(u) \overset{\mathcal{P}}{\leftrightarrow} \mathrm{res}_{\frac{\mathtt{D}}{\mathtt{C}}} \mathscr{I}(u) \,. \end{aligned} \tag{B.23}$$

It follows the transformation of the four kinematic regions and the corresponding expressions for the invariant in (4.170),

$$
\begin{aligned}
s_{234} > 0\,, s_{126} > 0 &\overset{\mathcal{P}}{\mapsto} s_{234} > 0\,, s_{126} > 0\,,\\
s_{234} < 0\,, s_{126} < 0 &\overset{\mathcal{P}}{\mapsto} s_{234} < 0\,, s_{126} < 0\,,\\
s_{234} > 0\,, s_{126} < 0 &\overset{\mathcal{P}}{\leftrightarrow} s_{234} < 0\,, s_{126} > 0\,.
\end{aligned}
\tag{B.24}
$$

Thus two of the kinematic regions of $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ for $\mathfrak{u}(2,2)$ are exchanged by the parity transformation. Each of the remaining two regions is parity invariant by itself.

As we argued in section 4.3.3.5, the Yangian invariant $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ with $c_1 = c_2 = c_3 = 0$ and $v_1 = v_2 = v_3$ agrees with the gluon amplitude $A^{(\text{tree})}_{6,3}$ merely in the kinematic region $s_{234}, s_{126} > 0$. We would like to show on principle grounds that the evaluation of the unitary Graßmannian integral (4.111) necessarily leads to the four kinematic regions of $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ in (4.170) despite the mismatch with $A^{(\text{tree})}_{6,3}$ in three of those. At the end of section B.2.2.2 we successfully employed the parity invariance to demonstrate the necessity of the two kinematic region of the Yangian invariant considered there. Proceeding analogously for $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$, we find that we cannot extend the expression in (4.170) for either of the regions $s_{234} > 0, s_{126} < 0$ and $s_{234} < 0, s_{126} > 0$ to the entire kinematic regime. The result would violate parity invariance according to (B.24). However, the expressions for the regions $s_{234} > 0, s_{126} > 0$ and $s_{234} < 0, s_{126} < 0$ could be extended to the entire domain without violating parity invariance. Consequently, the discrete parity symmetry is not sufficient to establish the necessity of the four kinematic regions of $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$. For this we would need a discrete symmetry group of $\Psi_{6,3}(\boldsymbol{\lambda},\overline{\boldsymbol{\lambda}})$ which does not leave any of the kinematic regions invariant. It would be very interesting to search for such a symmetry.

Let us remark that the parity transformation in (B.21) is part of a known discrete symmetry group of the amplitude $A^{(\text{tree})}_{6,3}$. This group includes a cyclic shift of the particle index $i \mapsto i+1$ and a reversal of the order of all particles, see e.g. [58] and recall the brief mention towards the end of section 1.3.4. These transformations act on the particle momenta and helicities. The transformation (B.21) is obtained from three shifts and one reversal. Most of the other transformations of this group are not straightforwardly implemented on the level of the unitary Graßmannian integral (4.111) because in our setting they would mix dual and ordinary sites.

B.3 Further Sample Invariants

In section 4.3.3 we evaluated the unitary Graßmannian integral (4.111) in spinor helicity-like variables for a number of sample Yangian invariants. We concentrated on those examples that are of help to understand how the scattering amplitudes of the introductory section 1.3 are related to our integral. Here we supplement the list of invariants by some further examples that do not directly serve this purpose but display other noteworthy features.

B.3.1 Two-Site Invariant for $\mathfrak{u}(2,2)$

We evaluated the unitary Graßmannian integral (4.111) with $(N,K) = (2,1)$ for the algebra $\mathfrak{u}(1,1)$ in section 4.3.3.2. The computation of this two-site invariant for $\mathfrak{u}(2,2)$ shows a peculiarity. From (4.111) with the parameterization of $U(1)$ in (4.37), the associated Haar measure (4.38) and the integrand $\mathscr{F}(\mathcal{C})$ from (4.31), we obtain

$$\Psi_{2,1}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}) = 2i\, \delta(P_{11})\delta_{\mathbb{C}}(P_{21})\lambda_1^1\overline{\lambda}_1^{-1}\left(-\frac{\lambda_1^1}{\lambda_1^2}\right)^{c_1-2}, \tag{B.25}$$

where we assumed $\lambda_1^2 \neq 0$ and the total momentum $P_{\alpha\dot{\alpha}}$ is defined in (4.115). Notice that (B.25) does not contain explicitly the momentum conserving delta function $\delta^{4|0}(P)$ from (4.116). However, the delta functions in (B.25) impose three real equations that imply the fourth equation $P_{22} = 0$ and thus implement the momentum conservation constraint (4.119). This is a special feature of two-particle kinematics in four-dimensional Minkowski space.

B.3.2 Four-Site Invariant for $\mathfrak{u}(2,2|4+0)$

In section 4.3.3.6 we computed the unitary Graßmannian integral (4.111) with four sites for the superalgebra $\mathfrak{u}(2,2|0+4)$. The notation $0+4 = r+s$ signifies the choice of the grading, cf. (2.1) and (2.32) where it affects the grouping of bosonic and fermionic oscillators. We made this particular choice in order to make contact with the superamplitudes reviewed in the introductory section 1.3.3. Here we investigate how the grading affects the form of the invariant. Computing the integral (4.111) for the four-site invariant with the superalgebra $\mathfrak{u}(2,2|4+0)$ leads to

$$\Psi_{4,2}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}, \boldsymbol{\theta}) = 8i\, \frac{\delta^{4|0}(P)\delta^{0|8}(\hat{Q})}{\overline{\langle 12\rangle}\,\overline{\langle 23\rangle}\,\overline{\langle 34\rangle}\,\overline{\langle 41\rangle}} \left(\frac{\langle 14\rangle}{\langle 34\rangle}\right)^{c_1}\left(\frac{\langle 12\rangle}{\langle 14\rangle}\right)^{c_2}\left(\frac{\langle 34\rangle\overline{\langle 34\rangle}}{\langle 14\rangle\overline{\langle 14\rangle}}\right)^{v_1-v_2}, \tag{B.26}$$

where the fermionic delta function is defined in (4.117). Interestingly, only the spinor brackets that remain in the undeformed case get complex conjugated compared to the $\mathfrak{u}(2,2|0+4)$ version in (4.174). Those factors involving the deformation parameters are identical for both gradings.

B.3.3 Four-Site Invariant for $\mathfrak{u}(2,2|2+2)$

Let us investigate the four-site invariant for yet another grading. The evaluation of the Graßmannian integral (4.111) with $(N,K) = (4,2)$ for the superalgebra $\mathfrak{u}(2,2|2+2)$ yields

$$\Psi_{4,2}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}, \boldsymbol{\theta}, \boldsymbol{\eta}) = 8i\, \frac{\delta^{4|0}(P)\delta^{0|4}(\hat{Q})\delta^{0|4}(Q)}{\langle 12\rangle\overline{\langle 23\rangle}\langle 34\rangle\overline{\langle 41\rangle}} \left(\frac{\langle 14\rangle}{\langle 34\rangle}\right)^{c_1}\left(\frac{\langle 12\rangle}{\langle 14\rangle}\right)^{c_2}\left(\frac{\langle 34\rangle\overline{\langle 34\rangle}}{\langle 14\rangle\overline{\langle 14\rangle}}\right)^{v_1-v_2}. \tag{B.27}$$

Note that $\langle 12\rangle\overline{\langle 23\rangle}\langle 34\rangle\overline{\langle 41\rangle} = \langle 12\rangle\overline{\langle 12\rangle}\langle 23\rangle\overline{\langle 23\rangle}$. Hence the combination of spinor brackets appearing in the undeformed Yangian invariant of the superalgebra $\mathfrak{u}(2,2|2+2)$ is real. Once more, those factors involving the deformation parameters are not affected by the choice of the grading, cf. the invariants (4.174) and (B.26).

We remark that the Yangian invariant corresponding to (B.27) in the oscillator basis is $|\Psi_{4,2}\rangle$ from (4.49). Each of its two building blocks $(k \bullet l)$ and $(k \circ l)$ is invariant under one of the two compact subalgebras in $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \subset \mathfrak{u}(2,2|2+2)$, as shown in (2.91). Similar $\mathfrak{su}(2|2)$ subalgebras play an important role to obtain asymptotic all-loop results in the planar $\mathcal{N} = 4$ SYM spectral problem, cf. [225] and recall section 1.2. This motivates our study of the Yangian invariant $\Psi_{4,2}(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}, \boldsymbol{\theta}, \boldsymbol{\eta})$ in spinor helicity variables with $2+2$ grading.

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